Some Solved Examples of Difference Equations

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We consider linear constant coefficient (LCC) $k$th-order difference equations, that is, equations of the form

\begin{equation}
    x_n - c_1 x_{n-1} - c_2 x_{n-2} \cdots - c_k x_{n-k} = \psi(n) \text{ for } n \geq k,
\end{equation}

where $c_i \in \mathbb{C}$ with $c_k \neq 0$ and $\psi(x)$ is usually called the forcing function. Here we consider the special situation in which the forcing function $\psi(n)$ has the form

\[
    \psi(n) = \gamma^n \cdot p(n)
\]

for $\gamma \in \mathbb{C}$ and certain polynomials $p(x) \in \mathbb{C}[x]$. We set $\lambda_1, \ldots, \lambda_t$ to be the distinct roots of the associated characteristic polynomial

\[
    ch(x) = x^k - c_1 x^{k-1} - \cdots - c_k.
\]

### 8.1 Simple roots ($t = k$)

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8. Worked Examples

**Solving (8.1) when there are no multiple roots**

When \( ch(x) \) has no multiple roots, the equation (8.1) has the solution

\[
(8.2) \quad x_n = \sum_{i=1}^{k} a_i \lambda_i^n + \gamma^n \cdot q(n),
\]

where \( a_1, \ldots, a_k \in \mathbb{C} \) and degree of the polynomial \( q(x) \in \mathbb{C}[x] \) satisfies

\[
(8.3) \quad \deg(q) = \begin{cases} 
\deg(p) & \text{if } \gamma \notin \{\lambda_1, \ldots, \lambda_k\} \\
\deg(p) + 1 & \text{if } \gamma \in \{\lambda_1, \ldots, \lambda_k\}.
\end{cases}
\]

All examples in this section will be second-order LCC equations with characteristic polynomial

\[ ch(x) = x^2 - x - 6 = (x - 3)(x + 2) \]

which has the simple roots \( \lambda_1 = 3, \lambda_2 = -2 \). Therefore, (8.2) and (8.3) become

\[
(8.4) \quad x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + \gamma^n \cdot q(n),
\]

where \( a_1, a_2 \in \mathbb{C} \) and degree of \( q(x) \) satisfies

\[
(8.5) \quad \deg(q) = \begin{cases} 
\deg(p) & \text{if } \gamma \neq 3, -2 \\
\deg(p) + 1 & \text{if } \gamma = 3 \text{ or } \gamma = -2.
\end{cases}
\]

**Example 8.1.1.** For the equation

\[ x_n = x_{n-1} + 6x_{n-2}, \]

\( \psi(x) \) is the zero polynomial which gives \( q(x) = 0 \) as well; there exist \( a_1, a_2 \in \mathbb{C} \) such that

\[
(8.6) \quad x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n.
\]

For initial conditions \( x_0, x_1 \) we therefore have

\[
(8.7) \quad x_0 = a_1 + a_2 \quad \text{and} \quad x_1 = 3a_1 - 2a_2.
\]

Multiplying the first equation by 3 and subtracting that from the second equation, we obtain

\[ x_1 - 3x_0 = -5a_2; \quad a_2 = \frac{3x_0 - x_1}{5}. \]
Inserting this value for \( a_2 \) into the first equation of (8.7) we obtain

\[
a_1 = x_0 - \frac{3x_0 - x_1}{5} = \frac{2x_0 + x_1}{5};
\]

the solution is

\[
x_n = \frac{2x_0 + x_1}{5}3^n + \frac{3x_0 - x_1}{5}(-2)^n.
\]

How do we obtain a closed form for the equation

\[
x_n = x_{n-1} + 6x_{n-2} + \psi(n)
\]

when \( \psi(x) \) is not the zero polynomial? It’s often fairly simple to compute a particular solution. Once a particular solution \( (v_n) \) is known then linearity allows us to use the general solution of the homogeneous equation to get the general solution

\[
x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + v_n.
\]

This procedure is used in the next examples.

**Example 8.1.2.** Consider the second order equation

(8.8) \[ x_n = x_{n-1} + 6x_{n-2} + 2^n, \]

with \( p(n) = 1 \) and \( \gamma = 2 \neq 3, -2 \). Then the general solution is

\[
x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + q(n) \cdot 2^n,
\]

for \( a_1, a_2 \in \mathbb{C} \) and \( q(x) \in \mathbb{C}[x] \) with \( \deg(q) = \deg(p) = 0; q(x) = c \in \mathbb{C} \) and we therefore obtain

\[
x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + c \cdot 2^n
\]

for \( a_1, a_2, c \in \mathbb{C} \). For \( a_1 = a_2 = 0 \) we obtain the solution \( v_n = c \cdot 2^n \), where

\[
v_{n-1} + 6v_{n-2} + 2^n = c2^{n-1} + 6c2^{n-2} + 2^n = 2^{n-1}(c + 3c + 2).
\]

Since \( v_n \) is the general term of a solution to (8.8),

\[
c \cdot 2^n = v_n = 2^{n-1}(4c + 2); \quad 2c = 4c + 2; \quad c = -1;
\]

that is, \( v_n = -2^n \) and

(8.9) \[ x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n - 2^n, \] for some \( a_1, a_2 \in \mathbb{C} \).

For initial values \( x_0, x_1 \) we therefore obtain

(8.10) \[ x_0 = a_1 + a_2 - 1 \] and \( x_1 = 3a_1 - 2a_2 - 2, \)
which we want to solve for $a_1, a_2$ as in the previous example. Multiplying
the first equation by 2 and adding it to the second, we have

$$2x_0 + x_1 = 5a_1 - 4 \quad ; \quad a_1 = \frac{2x_0 + x_1 + 4}{5}.$$  

Substituting this value of $a_1$ into the first equation of (8.10) we obtain

$$a_2 = x_0 - a_1 + 1 = x_0 - \frac{2x_0 + x_1 + 4}{5} + 1 = \frac{3x_0 - x_1 + 1}{5},$$

and (8.9) becomes

$$x_n = \frac{2x_0 + x_1 + 4}{5} \cdot 3^n + \frac{3x_0 - x_1 + 1}{5} \cdot (-2)^n - 2^n.$$  

Checking the next term of the sequence we have

$$x_2 = \frac{2x_0 + x_1 + 4}{5} \cdot 9 + \frac{3x_0 - x_1 + 1}{5} \cdot 4 - 4 = x_1 + 6x_0 + 4,$$

as required by the recurrence.

**Example 8.1.3.** The second-order equation

(8.11) \quad x_n = x_{n-1} + 6x_{n-2} + n2^n

has $p(n) = n$ (with $\deg(p) = 1$) and $\gamma = 2 \neq 3, -2$; $v_n = (an + b) \cdot 2^n$ for

constants $a, b$ and

$$v_{n-1} + 6v_{n-2} + n2^n - v_n = (an - a + b)2^{n-1} + 6(an - 2a + b) \cdot 2^{n-2} + n2^n - (an + b)2^n$$

$$= 2^{n-1}((an - a + b) + 3(an - 2a + b) + 2n - 2(an + b))$$

$$= 2^{n-1}((2a + 2)n + (-7a + 2b)),$$

which equals zero for all $n \geq 0$ when $2a + 2 = 0$ and $-7a + 2b = 0$; $a = -1$

and $2b = 7a = -7$. Therefore, $v_n = -(n + \frac{7}{2})2^n$ is a particular solution of

(8.11) and the general solution has the form

$$x_n = a_1 3^n + a_2 (-2)^n - (n + \frac{7}{2})2^n \quad \text{for} \quad a_1, a_2 \in \mathbb{C}.$$  

For any initial values $x_0, x_1$ from this we obtain:

$$x_0 = a_1 + a_2 - \frac{7}{2} \quad \text{and} \quad x_1 = 3a_1 - 2a_2 - 9.$$  

Simultaneously solving this system of equations, we obtain

$$a_1 = \frac{2x_0 + x_1 + 16}{5} \quad \text{and} \quad a_2 = \frac{3x_0 - x_1 + \frac{3}{2}}{5};$$
\[ x_n = \frac{2x_0 + x_1 + 16}{5} \cdot 3^n + \frac{3x_0 - x_1 + \frac{3}{2}}{5} \cdot (-2)^n - \left( n + \frac{7}{2} \right)2^n. \]

Verifying this calculation for \( n = 2 \):
\[ x_2 = \frac{18x_0 + 9x_1 + 144}{5} + \frac{12x_0 - 4x_1 + 6}{5} - \left( 2 + \frac{7}{2} \right)4 = 6x_0 + x_1 + 8, \]
as required.

**Example 8.1.4.** For
\[ x_n = x_{n-1} + 6x_{n-2} + 3^n, \]
\( p(x) = 1 \) and \( \gamma = 3 = \lambda_1 \), from which we obtain
\[ x_n = a_1 3^n + a_2(-2)^n + 3^n \cdot q(n) \]
for polynomial \( q(x) \) with \( \text{deg}(q) = 1 \). Therefore,
\[ x_n = a_1 3^n + a_2(-2)^n + 3^n(a + bn) = (a_1 + a)3^n + a_2(-2)^n + bn3^n, \]
and we can choose \( v_n = bn \cdot 3^n \) for \( b \in \mathbb{C} \). (Note how the constant term of \( q(x) \) has been absorbed into the earlier coefficient of \( 3^n \).) Then
\[ v_{n-1} + 6v_{n-2} + 3^n - v_n = 3^{n-1}b(n - 1 + 2n - 4 - 3n) + 3^n = 3^{n-1}(-5b + 3); \]
from this, \( b = \frac{3}{5} \) and \( v_n = \frac{3}{5}n3^{n+1} \). Therefore,
\[ x_0 = a_1 + a_2 \quad \text{and} \quad x_1 = 3a_1 - 2a_2 + \frac{9}{5}, \]
which gives
\[ a_1 = \frac{2x_0 + x_1 - \frac{9}{5}}{5}; \quad a_2 = \frac{3x_0 - x_1 + \frac{9}{5}}{5}. \]

Hence,
\[ x_n = \frac{1}{5} \left( (2x_0 + x_1 - \frac{9}{5})3^n + (3x_0 - x_1 + \frac{9}{5})(-2)^n + n3^{n+1} \right). \]

**Example 8.1.5.** For
\[ x_n = x_{n-1} + 6x_{n-2} + n(-2)^n \]
\( p(x) = 1 \) and \( \gamma = -2 = \lambda_2 \), from which we obtain
\[ x_n = a_1 3^n + a_2(-2)^n + (-2)^n \cdot q(n) \]
for \( q(x) \) with \( \deg(q) = 2 \). Therefore, again absorbing the constant term of \( q(x) \), there exist \( b, c \in \mathbb{C} \) such that

\[
x_n = a_1 3^n + a_2 (-2)^n + (-2)^n (bn + cn^2);
\]

(8.12) \[
x_n = a_1 3^n + a_2 (-2)^n + v_n,
\]

where \( v_n = (cn^2 + bn)(-2)^n \). Then

\[
v_{n-1} + 6v_{n-2} + (-2)^n - v_n
\]

\[
= (-2)^n (-1)(c(n-1)^2 + b(n-1) - 3(c(n-2)^2 + b(n-2)) - 2n + 2(cn^2 + bn))
\]

\[
= (-2)^n (-1)(10c - 2)n + (5b - 11c)
\]

equals zero for all \( n \geq 0 \) when

\[
10c = 2 \text{ and } 5b = 11c;
\]

that is, \( c = \frac{1}{5}, b = \frac{11}{5} \) and \( v_n = (\frac{11}{5} + \frac{5}{3}n)(-2)^n \). Therefore, putting \( v_0 = 0 \) and \( v_1 = \frac{32}{5} \) in (8.12) we obtain

\[
x_0 = a_1 + a_2 \text{ and } x_1 = 3a_1 - 2a_2 - \frac{32}{5}
\]

and from this

\[
a_1 = \frac{2x_0 + x_1 + \frac{32}{5}}{5}; \quad a_2 = \frac{3x_0 - x_1 - \frac{32}{5}}{5}.
\]

Hence,

\[
x_n = \frac{1}{5} \left( \left(2x_0 + x_1 + \frac{32}{5}\right)3^n + \left(3x_0 - x_1 - \frac{32}{5} + \frac{11}{25}n + \frac{1}{5}n^2\right)(-2)^n \right).
\]

8.2 Multiple roots \( (t < k) \)

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8.2 Multiple roots \((t < k)\)

Solution to (8.1) when there is one multiple root

When \(ch(x)\) has \(\lambda_1 = \lambda_2\) and all of \(\lambda_3, \ldots, \lambda_k\) are distinct, the equation (8.1) has the solution

\[
x_n = (a_1 n + a_2)\lambda_1^n + \sum_{i=3}^{k} a_i \lambda_i^n + \gamma^n \cdot q(n),
\]

where \(a_1, \ldots, a_k \in \mathbb{C}\) and the degree of the polynomial \(q(x) \in \mathbb{C}[x]\) satisfies

\[
\text{deg}(q) = \begin{cases} 
\text{deg}(p) & \text{if } \gamma \notin \{\lambda_1, \ldots, \lambda_t\} \\
\text{deg}(p) + 1 & \text{if } \gamma \in \{\lambda_3, \ldots, \lambda_t\} \\
\text{deg}(p) + 2 & \text{if } \gamma = \lambda_1 
\end{cases}
\]

The first five examples in this section are second-order linear constant coefficient recurrence equations with characteristic polynomial

\[ch(x) = x^2 - 4x + 4 = (x - 2)^2,\]

which has the double root \(\lambda_1 = \lambda_2 = 2\). Therefore, (8.13) and (8.14) become

\[
x_n = (a_1 n + a_2)2^n + \gamma^n \cdot q(n),
\]

where \(a_1, a_2 \in \mathbb{C}\) and degree of the polynomial \(q(x) \in \mathbb{C}[x]\) satisfies

\[
\text{deg}(q) = \begin{cases} 
\text{deg}(p) & \text{if } \gamma \neq 2 \\
\text{deg}(p) + 2 & \text{if } \gamma = 2 
\end{cases}
\]

**Example 8.2.1.** For the equation

\[x_n = 4x_{n-1} - 4x_{n-2},\]

\(\psi(x)\) is the zero polynomial which gives \(q(x) = 0\); there exist \(a_1, a_2 \in \mathbb{C}\) such that

\[
x_n = (a_1 n + a_2)2^n.
\]

For initial conditions \(x_0, x_1\) we can compute \(a_2 = x_0\) and \(a_1 = \frac{x_1 - 2x_0}{2}\), which gives the general term

\[x_n = ((x_1 - 2x_0)n + 2x_0)2^{n-1}.
\]
As we observed in the previous section when we considered the case of simple roots, if \( (v_n) \) is any particular solution to the given equation then the general solution has the form
\[
x_n = (a_1 n + a_2)2^n + v_n,
\]
for some \( a_1, a_2 \in \mathbb{C} \).

**Example 8.2.2.** Consider the second order equation
\[
(8.18) \quad x_n = 4x_{n-1} - 4x_{n-2} + 3^n,
\]
with \( p(n) = 1 \) and \( \gamma = 3 \neq 2 \). Then the general solution is
\[
x_n = (a_1 n + a_2)2^n + q(n) \cdot 3^n,
\]
for \( a_1, a_2 \in \mathbb{C} \) and \( q(x) \in \mathbb{C}[x] \) with \( \deg(q) = \deg(p) = 0; q(x) = c \in \mathbb{C} \) and
\[
x_n = (a_1 n + a_2)2^n + c3^n.
\]
For \( a_1 = a_2 = 0 \) we obtain the solution \( v_n = c \cdot 3^n \), where
\[
4v_{n-1} - 4v_{n-2} + 3^n - v_n = 3^{n-2}(12c - 4c + 9 - 9c) = 3^{n-2}(9 - c),
\]
which equals zero for \( c = 9 \). From this we obtain the particular solution with general term \( v_n = 3^{n+2} \) and
\[
(8.19) \quad x_n = (a_1 n + a_2)2^n + 3^{n+2}, \text{ for some } a_1, a_2 \in \mathbb{C}.
\]
For initial conditions \( x_0, x_1 \) computation gives
\[
x_n = \left(\frac{x_1 - 2x_0 - 9}{2} + (x_0 - 9)\right)2^n + 3^{n+2}.
\]

**Example 8.2.3.** The second-order equation
\[
(8.20) \quad x_n = 4x_{n-1} - 4x_{n-2} + n3^n
\]
has \( \deg(p) = 1 \) and \( \gamma = 3 \neq 2 \). Therefore, we use \( v_n = (an + b) \cdot 3^n \) for constants \( a, b \) and have
\[
4v_{n-1} - 4v_{n-2} + n3^n - v_n
\]
\[
= 3^{n-2}(12(an - a + b) - 4(an - 2a + b) + 9n - 9(an + b))
\]
\[
= 3^{n-2}((9 - a)n - (4a + b)),
\]
which equals zero when \( a = 9 \) and \( b = -4a = -36 \). Therefore, \( v_n = (n - 4)3^{n+2} \) is a particular solution of (8.20) and the general solution has the form
\[
x_n = (a_1 n + a_2)2^n + (n - 4)3^{n+2}.
\]
For initial values \( x_0, x_1 \) we obtain:
\[
x_n = \left(\frac{x_1 - 2x_0 + 9}{2} + (x_0 + 36)\right)2^n + (n - 4)3^{n+2}.
\]
Example 8.2.4. The second-order equation

\[(8.21) \quad x_n = 4x_{n-1} - 4x_{n-2} + 2^n\]

has \(\text{deg}(p) = 0\) and \(\gamma = 2 = \lambda_1\). Therefore, \(\text{deg}(p) = 2\) and let \(v_n = an^22^n\) for \(a \in \mathbb{C}\). Then

\[
4v_{n-1} - 4v_{n-2} + 2^n - v_n
= 2^n(2a(n - 1)^2 - a(n - 2)^2 + 1 - an^2)
= 2^n(-2a + 1),
\]

implying for \(a = \frac{1}{2}\), the sequence with general term \(v_n = n^22^{n-1}\) is a particular solution of (8.21). The general solution has the form

\[x_n = (n^2 + a_1n + a_2)2^{n-1}.\]

For initial values \(x_0, x_1\) we obtain:

\[x_n = (n^2 + (x_1 - 1 - 2x_0)n + 2x_0)2^{n-1}.
\]

Example 8.2.5. The second-order equation

\[(8.22) \quad x_n = 4x_{n-1} - 4x_{n-2} + n2^n\]

has \(\text{deg}(p) = 1\) and \(\gamma = 2 = \lambda_1\). Therefore, \(\text{deg}(p) = 3\); \(v_n = (an^3 + bn^2)2^n\) for \(a, b \in \mathbb{C}\). Then

\[
4v_{n-1} - 4v_{n-2} + n2^n - v_n
= 2^n(2(a(n - 1)^3 + b(n - 1)^2) - (a(n - 2)^3 + b(n - 2)^2) + n - (an^3 + bn^2))
= 2^n[-6an - 2b + 6a + n] = 2^n[n(1 - 6a) + 2(3a - b)],
\]

which equals zero when \(a = \frac{1}{3}\) and \(b = 3a = \frac{1}{2}\); \(v_n = (\frac{1}{3}n^3 + n^2)2^{n-1}\) is a particular solution of (8.22). The general solution therefore has the form

\[x_n = (\frac{1}{3}n^3 + n^2 + a_1n + a_2)2^{n-1}.
\]

Initial values \(x_0, x_1\) give

\[x_n = (\frac{1}{3}n^3 + n^2 + (x_1 - 2x_0 - \frac{4}{3})n + 2x_0)2^{n-1}.
\]

Since each of the last five examples had only one (double) root, it was impossible to be in the middle alternative of (8.14). The next example illustrates that case.
Example 8.2.6. The second-order equation

\[(8.23) \quad x_n = 4x_{n-1} - 5x_{n-2} + 2x_{n-3} + 2^n\]

has \(\text{deg}(p) = 0\) and \(\gamma = 2\). Since the characteristic polynomial is

\[ch(x) = x^3 - 4x^2 + 5x - 2 = (x - 1)^2(x - 2),\]

\(\gamma\) is a simple root of \(ch(x)\) and the general solution is

\[x_n = (a_1 n + a_2) + a_3 2^n + 2^n \cdot q(n),\]

where \(a_1, a_2, a_3 \in \mathbb{C}\) and \(\text{deg}(q) = 1\). Setting \(v_n = an2^n\),

\[4v_{n-1} - 5v_{n-2} + 2v_{n-3} + 2^n - v_n = 2^{n-2}(8a(n - 1) - 5a(n - 2) + a(n - 3) + 4 - 4an) = 2^{n-2}(4 - a);\]

\[v_n = 4n2^n = n2^{n+2}\] is a particular solution of \((8.23)\), and the general solution has

\[x_n = (a_1 n + a_2) + a_3 2^n + n2^{n+2}.\]

For initial values \(x_0, x_1, x_2\) we obtain

\[x_0 = a_2 + a_3; \quad x_1 = a_1 + a_2 + 2a_3 + 8; \quad x_2 = 2a_1 + a_2 + 4a_3 + 32;\]

which implies

\[x_n = (-2x_0 + 3x_1 - x_2 + 8)n + (2x_1 - x_2 + 16)\]

\[+(x_0 - 2x_1 + x_2 - 16)2^n + n2^{n+2}.\]