Homework # 5

Divide and Conquer Recurrences

For the following algorithms given in schematic form, write down a divide-and-conquer recurrence relation for the run time, and solve for the run time in Big Theta form.

(a)

\[ \textsc{Procedure} \ \textsc{Mult}(a, b, n) \]

Split a into a0, a1, and b into b0, b1
\[
\textsc{Mult}(a0, b0, n/2): \textsc{Mult}(a1, b1, n/2): \textsc{Mult}(a1 + a0, b1 + b0, n/2)
\]

\text{Combine the results of the subproblems}

If Split and Combine have Θ(n) run time then Mult has runtime

(b)

\[ \textsc{Function} \ \textsc{Fool}(x, n) \]

Split X into X1 and X2
\[
Y1 = \textsc{Fool}(X1, n/2)
Y2 = \textsc{Fool}(X2, n/2)
\]

\text{Combine(Y1, Y2)}

If Split and Combine are both Θ(n) then Fool has runtime

(c)

\[ \textsc{Function} \ \textsc{Mess}(x, n) \]

Split X into Y1, Y2, Y3
\text{Test(Y1, Y2)}

depending on the outcome of Test
\[
\text{Do} \ \textsc{Mess}(Y1, n/3)
\text{or Mess}(Y2, n/3)
\text{or Mess}(Y3, n/3)
\]
If SPLIT takes $\Theta(n^2)$ and TEST takes $\Theta(1)$ then MESS has runtime

(d) If SPLIT takes $\Theta(\log n)$ and TEST takes $\Theta(1)$ then MESS from (c) has runtime

(d)

\begin{verbatim}
PROCEDURE MULT( a, b, n)
    Split a into a0, a1, and b into b0, b1
    MULT( a0, b0, n/2)
    MULT( a1, b1, n/2)
    MULT( a1 + a0, b1 + b0, n/2)
    COMBINE the results of the subproblems
\end{verbatim}

If COMBINE has $\Theta(n)$ run time then MULT has runtime

Construction of a Tree from a Degree List

A graph is a set of vertices (singular: vertex) and a set of edges, each edge connecting two vertices together. The degree of a vertex is the number of edges which connect to the vertex. Said another way, the degree of a vertex is the number of vertices adjacent to the given vertex.

In a graph with $n$ vertices $v_1, v_2, \ldots, v_n$, let $d_1, d_2, \ldots, d_n$ be the respective degrees of those vertices. The degree of a vertex $v_i$ in a connected graph is bounded, $1 \leq d_i < n$.

A tree is a connected graph with $n$ vertices and $n - 1$ edges. Since each edge connects to two vertices, it is easy to see that in a tree,\[ \sum_{i=1}^{n} d_i = 2(n - 1) \]

because each of the $n - 1$ edges contribute 2 to the sum of degrees. We want to turn this around and show how to find a tree, given a set of degrees which satisfy the above summation constraint. This leads to the following Tree Construction Problem.
Given a set of positive integers, \(d_1, d_2, \ldots, d_n\), such that \(\sum_{i=1}^{n} d_i = 2(n-1)\), construct a tree with \(n\) vertices in which \(\text{degree}(v_i) = d_i\).

Problems

1. Write an inductive proof of
   
   If \(\sum_{i=1}^{n} d_i = 2(n-1)\), where \(d_1, d_2, \ldots, d_n\) are positive integers, then there is a tree with \(n\) vertices in which \(\text{degree}(v_i) = d_i\).

2. What are you inducting on? What is the inductive variable?

3. State your inductive hypothesis as a function of your inductive variable.

4. What is the BASE case?

5. Prove the BASE case.

6. Carry out the INDUCTIVE STEP of the proof.
   
   This may involve some algebraic manipulation.

7. Use your inductive proof to design a divide and conquer algorithm for the tree construction problem.

8. Program your algorithm and demonstrate by examples that your program produces correct output.

9. Write and solve a difference equation for the running time of your program.

10. Run your program for different problem sizes and plot the running times to show that your program behaves according to your analysis in \#9. Remember to choose the appropriate graph type.

Hints

The simplest way to represent a tree is by a parent array, in which \(\text{PARENT}[i] = j\) if and only if \(v_j\) is the parent of \(v_i\) in the tree hierarchy. Of course the root, \(v_r\), will have no parent, which can be represented by \(\text{PARENT}[r] = 0\).

A vertex at the end of the hierarchy in a directed tree, which has no children vertices, is called a leaf. What can you say about the degrees of leaf vertices?