1. Let \( x(t) = e^{-j\frac{\pi}{3}t} + \sin\left(\frac{\pi}{3}t\right) \)

   (a) Is \( x(t) \) periodic? If so, find its fundamental period. If not, explain why? (10 points)

   (b) Let \( y[n] = x(2n) \), i.e., \( y[n] \) is the sampled signal of \( x(t) \) at every interval \( T = 2 \). Is \( y[n] \) periodic? If so, find its fundamental period. If not, explain why? (10 points)

\[
x(t + T_p) = e^{-j\frac{\pi}{3}(t+T_p)} + \sin\left(\frac{\pi}{3}(t + T_p)\right) = e^{-j\frac{\pi}{3}t + \frac{\pi}{3}T_p} + \sin\left(\frac{\pi}{3}t + \frac{\pi}{3}T_p\right)
\]

\( x(t) \) is periodic iff \( \frac{\pi}{2}T_p = k2\pi \) and \( \frac{\pi}{3}T_p = l2\pi \) \( \Rightarrow \) \( k = \frac{3}{2} l \)

\( \Rightarrow \) \( x(t) \) is periodic. Its fundamental period is \( T_p = 12 \) (when \( k=3, l=2 \))

b) \( y[n] = x(2n) = x(nT_s) \)

\( y[n+Np] = x((n+Np)T_s) = x(nT_s+NpT_s) \)

\( \Rightarrow \) \( y[n] \) is periodic iff \( NpT_s = T_p \) \( \Rightarrow \) \( Np = \frac{T_p}{T_s} = \frac{12}{2} = 6 \)
2. For each following systems, determine whether they are time-invariant, BIBO stable, causal. Provide reasons for each answer. (40pts)

(a) \( y[n] = \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i x[i+1] \)

\[
y[n - n_0] = \sum_{i=-2}^{n-n_0-1} \left( \frac{1}{2} \right)^i x[i + 1]
\]

\[
y'[n] = \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i x[i - n_0 + 1]
\]

Let \( j = i - n_0 \)

\[
y'[n] = \sum_{j=-2-n_0}^{n-n_0-1} \left( \frac{1}{2} \right)^{j+n_0} x[j + 1]
\]

=> time variant

With bounded Input: \( |x[n]| \leq M_x \)

\[
|y[n]| = \left| \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i x[i + 1] \right| \leq \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i |x[i + 1]| \leq \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i M_x = M_x \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i
\]

\[
= M_x \sum_{j=0}^{n+1} \left( \frac{1}{2} \right)^{j-2} = 4M_x \sum_{j=0}^{n+1} \left( \frac{1}{2} \right)^j = 4M_x \frac{1 - \left( \frac{1}{2} \right)^{n+2}}{1 - \frac{1}{2}} = M_y \text{ (finite with all } n)\]

\( \Rightarrow \) system is BIBO stable

\[
y[n] = \sum_{i=-2}^{n-1} \left( \frac{1}{2} \right)^i x[i + 1] = \left( \frac{1}{2} \right)^2 x[-1] + \cdots + \left( \frac{1}{2} \right)^{n-1} x[n]
\]

System depends only on past and current values of the input \( \Rightarrow \) causal.
(b) \( y(t) = \int_{t-1}^{t+1} e^{u-t} x(u-1) du \)

\[
y'(t) = \int_{t-1}^{t+1} e^{u-t} x(u-T-1) du
\]

\[
y(t-T) = \int_{t-T-1}^{t-T+1} e^{u-t+T} x(u-1) du
\]

Let \( v = u+T \Rightarrow dv = du \)

\[
y(t-T) = \int_{t-1}^{t+1} e^{v-t} x(v-T-1) dv = y'(t)
\]

\( \Rightarrow \) system id time invariant

with bounded input: \( |x(t)| \leq M_x \)

\[
|y(t)| = \left| \int_{t-1}^{t+1} e^{u-t} x(u-1) du \right| \leq \int_{t-1}^{t+1} e^{u-t} |x(u-1)| du \leq \int_{t-1}^{t+1} e^{u-t} M_x du = M_x \int_{t-1}^{t+1} e^{u-t} du
\]

Let \( v = u-t+1 \Rightarrow du = dv \)

\[
\int_{t-1}^{t+1} e^{u-t} du = \int_{0}^{2} e^{v-t+1-t} dv = \int_{0}^{2} e^{v-1} dv = e^{v-1} \bigg|_{0}^{2} = e - e^{-1}
\]

\( \Rightarrow |y(t)| \leq (e - e^{-1})M_x \) system is BIBO stable

Let \( v = u-t+1 \Rightarrow du = dv \)

\[
y(t) = \int_{0}^{2} e^{v-1} x(v + t - 2) dv
\]

Since \( v \ [0,2] \Rightarrow v+t-2 \ [t-2,t] \Rightarrow system depends only on the past and the current value of x(t) \)

\( \Rightarrow \) system is causal.
3. Given $x(t)$ and $h(t)$ below, find and carefully sketch the output $y(t) = x(t) * h(t)$. Make sure you label all the points carefully (10pts).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

- $t < 0 \implies y(t) = 0$
- $0 \leq t \leq 1$
  $$y(t) = x(t) * h(t) = \int_{0}^{t} d\tau = t$$
- $1 \leq t \leq 2$
  $$y(t) = x(t) * h(t) = \int_{t-1}^{1} d\tau = 2 - t$$
- $2 < t \leq 3$
  $$y(t) = x(t) * h(t) = 1$$
- $t > 3 \implies y(t) = 0$