4. **Markov Game**

Your friend and you play a board game. Each takes turn to play one of the two possible moves. Let call them $L$ and $R$. Each of you comes up with your own strategy based on the immediate previous move of your opponent. Specifically, you will use the same move as your friend’s move with probability $p$ and $1-p$ otherwise. For example, if your friend’s last move is $L$, you will move $L$ with probability $p$ and move $R$ with probability $1-p$. Similarly, your friend will use the same move as yours with probability $q$ and $1-q$ otherwise. Let $X_1, X_2, X_3 \ldots$ denote a sequence of moves from you and your friend, $X_i \in \{L, R\}$.

(a) Is $X_1, X_2, X_3 \ldots$ a stationary random process? Provide a mathematical justification.

(b) Show that $X_1, X_3, X_5 \ldots$ is an ergodic process.

(c) Compute the stationary distribution of $X_1, X_3, X_5 \ldots$.

(d) Compute the entropy rate $R$ of the sequence $X_1, X_3, X_5 \ldots$.

(e) Does the entropy rate $H(X) = X_1, X_2, X_3 \ldots$ exist? If so, compute $H(X)$. If not, show why.

(f) Suppose you want to produce the sequence of moves of the game that when compressed has smallest expected length per move. You cannot control your friend’s move, but you can control yours. Suggest a strategy for doing so. What is the expected length in term of $q$?