9-2 THE ADJOINT NETWORK

In Chap. 2, we derived the following forms of Tellegen’s theorem:

\[ \sum_{k=1}^{N} v_k j_k = v^T j' = 0 \quad \sum_{k=1}^{N} v'_k j_k = v'^T j = 0 \quad (9-10) \]

[See Eqs. (2-36) and (2-37).] Here, the \( v_k \) and \( j_k \) are the branch voltages and currents of a network \( N \), with associated reference directions being used (Fig. 2-4). Similarly, \( v'_k \) and \( j'_k \) are the branch variables of a second network \( N' \), also with associated directions. Since \( N \) and \( N' \) have the same graph, i.e., the same topological configuration, they have the same number of nodes and branches; corresponding branches are incident on corresponding nodes. Furthermore, the node and branch numbering and the reference directions are the same for the two circuits (see, for example, the networks in Figs. 2-4 and 2-5).

Consider now the three circuits shown in Fig. 9-4. In the figure, \( N \) is the physical network whose sensitivities we have to calculate. \( N_\Delta \) is the incremented network in which all variable elements \( p_i \) have been replaced by their incremented values \( p_i + \Delta p_i \). Finally, \( \tilde{N} \) is the adjoint network, to be discussed. All three circuits satisfy pairwise the conditions described above for \( N \) and \( N' \). Thus, all three must have the same topology, same node and branch numbering, reference directions, etc. Furthermore, we postulate that the generators (independent sources) of \( \tilde{N} \) are located in the same branches as those of \( N \) and \( N_\Delta \). The types of these sources (voltage or current) must also be the same; however, their values may be different.

Applying Tellegen’s theorem to \( N \) and \( \tilde{N} \) gives

\[ v^T j = 0 \quad \tilde{v}^T j = 0 \quad (9-11) \]

and so also

\[ v^T j - \tilde{v}^T j = 0 \quad (9-12) \]

Applying Tellegen’s theorem to \( N_\Delta \) and \( \tilde{N} \) gives

\[ (v + \Delta v)^T j = 0 \quad \tilde{v}^T (j + \Delta j) = 0 \quad (9-13) \]

where \( \Delta j \) and \( \Delta v \) indicate that the currents and voltages of \( N_\Delta \) will differ from those of \( N \), due to the changes in the \( p_i \). From (9-13),

\[ (v + \Delta v)^T j - \tilde{v}^T (j + \Delta j) = 0 \quad (9-14) \]

Subtracting (9-12) from (9-14) gives

DIFF. TELLEGEN’S THEOREM \[ \Delta v^T j - \tilde{v}^T \Delta j = 0 \quad (9-15) \]
sometimes called the differential Tellegen's theorem. In scalar form, Eq. (9-15) becomes

\[ \sum_{\text{all branches}} (\Delta j_k - \Delta i_k) = 0 \]  

(9-16)

Here, the \( \Delta v_k \) and \( \Delta j_k \) are the results of the changes in the variable circuit parameters from \( p_i \) to \( p_i + \Delta p_i \), \( i = 1, 2, \ldots, n \).

We shall next show how to determine the sensitivities of the simple circuit of
Fig. 9-1a from (9-16). The circuit is redrawn in Fig. 9-5a with the notation of Fig. 9-4. Note, in particular, that the open-circuit output “branch” has been replaced by an equivalent 0-A current source. This does not represent any change, but it makes the following discussions somewhat easier. The adjoint network is shown in Fig. 9-5b. We only know the configuration and the location of its sources; hence, the other branches are represented by black boxes.

Applying the differential Tellegen’s theorem to the two circuits of Fig. 9-5a and b gives

\[ (J_{e_1} \Delta v_1 - \hat{e}_1 \Delta J_{e_1}) + (J_{i_1} \Delta v_1 - \hat{v}_1 \Delta J_{i_1}) + (J_{i_2} \Delta v_2 - \hat{v}_2 \Delta J_{i_2}) + (\hat{i}_1 \Delta v_{i_1} - \hat{v}_{i_1} \Delta i_1) = 0 \quad (9-17) \]

Our purpose is to bring (9-17) into the form of (9-4). Since \( v_0 \equiv v_{i_1} \), we have to (1) introduce \( \Delta R_1 \) and \( \Delta R_2 \) into (9-17) and (2) suppress all terms except those containing \( \Delta R_1, \Delta R_2, \) and \( \Delta v_{i_1} \). These objectives will be achieved by choosing the structure and element values of \( \hat{N} \), hitherto unspecified, in an appropriate way.

Consider the first two terms on the left-hand side of (9-17). Since \( e_1 \) is a

We want
\[ \Delta v_{i_1} \equiv S_{R_1} \Delta R_1 + S_{R_2} \Delta R_2 \]
\[ \Delta v_0 \equiv \frac{\partial v_0}{\partial R_1} \Delta R_1 + ... \]
generator which we assume to be unaffected by tolerances, \( \Delta e_1 \equiv 0 \). Since the second term does not contain \( \Delta R_1 \), \( \Delta R_2 \), or \( \Delta v_i \), we suppress it by choosing \( \hat{e}_1 \equiv 0 \) in \( \hat{N} \) (Fig. 9-5c).

In the second two terms, \( \Delta R_1 \) can be introduced through the branch relations of the original network \( N \):

\[
v_1 = R_1 j_1
\]

and of the incremented network \( N_A \):

\[
v_1 + \Delta v_1 = (R_1 + \Delta R_1)(j_1 + \Delta j_1)
\]

Subtracting (9-18) from (9-19) gives

\[
\Delta R_1, \Delta j_1 \rightarrow 0 \quad \Delta v_1 = R_1 \Delta j_1 + \Delta R_1 j_1 + \Delta R_1 \Delta j_1
\]

If all increments are small, \( \Delta R_1, \Delta j_1 \) is a second-order small term and can be neglected. Hence

\[
\hat{j}_1 \Delta v_1 - \hat{v}_1 \Delta j_1 \approx \hat{j}_1 (R_1 \Delta j_1 + \Delta R_1 j_1) - \hat{v}_1 \Delta j_1 = j_1 \hat{j}_1 \Delta R_1 + (\hat{j}_1, R_1 - \hat{v}_1) \Delta j_1
\]

The first term on the right-hand side of (9-21) contains \( \Delta R_1 \) and should hence be retained. The second term can be suppressed by making

\[
\hat{v}_1 = R_1 \hat{j}_1
\]

which implies that this branch of \( \hat{N} \) should contain a resistor \( R_1 \) (Fig. 9-5c). Then the right-hand side of (9-21) becomes \( j_1 \hat{j}_1 \Delta R_1 \).

An analog derivation shows that

\[
\hat{j}_2 \Delta v_2 - \hat{v}_2 \Delta j_2 = j_2 \hat{j}_2 \Delta R_2
\]

provided that

\[
\hat{v}_2 = R_2 \hat{j}_2
\]

i.e., provided that this branch of \( \hat{N} \) contains the resistor \( R_2 \) (Fig. 9-5c).

Finally, in the last two terms of (9-17), \( \Delta i_i \equiv 0 \). The term \( \hat{i}_i \Delta v_i \) must be kept. If, for convenience, we choose \( \hat{i}_i = 1 \) A (Fig. 9-5c), then (9-17) becomes finally

\[
j_1 \hat{j}_1 \Delta R_1 + j_2 \hat{j}_2 \Delta R_2 + \Delta v_i = 0
\]

\[
\Delta v_0 \equiv \Delta v_i = -j_1 \hat{j}_1 \Delta R_1 - j_2 \hat{j}_2 \Delta R_2
\]

A comparison of (9-26) and (9-4) shows immediately that the desired sensitivities are

\[
\frac{\partial v_0}{\partial R_1} = \frac{\partial v_i}{\partial R_1} = -j_1 \hat{j}_1 \quad \frac{\partial v_0}{\partial R_2} = \frac{\partial v_i}{\partial R_2} = -j_2 \hat{j}_2
\]

Here, \( j_1 \) and \( j_2 \) are the currents through \( R_1 \) and \( R_2 \) in \( N \) (Fig. 9-5a). By inspection,

\[
j_1 = j_2 = \frac{e_1}{R_1 + R_2} \approx 0.01333 \text{ A}
\]

\[\hat{j} \text{ proportional to transmission from } R_i \text{ to output}\]
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\[ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \]

\[ V_B = \frac{R}{R} \frac{R}{R} \]

\[ \Delta R = \begin{bmatrix} \Delta R_{11} & \Delta R_{12} \\ \Delta R_{21} & \Delta R_{22} \end{bmatrix} \]

transimpedance transresistance

\[ R_{in} j \omega = \frac{R}{R} \frac{R}{R} \]

CCVS
The currents \( \hat{j}_1 \) and \( \hat{j}_2 \) are the corresponding currents in the adjoint network \( \hat{N} \). From Fig. 9-5c, also by inspection,

\[
\hat{v}_2 = -\hat{j}_1 \frac{R_1 R_2}{R_1 + R_2} \approx -33.33 \text{ V}
\]

and

\[
\hat{j}_1 = -\frac{\hat{v}_2}{R_1} \approx 0.6667 \text{ A} \quad \hat{j}_2 = \frac{\hat{v}_2}{R_2} \approx -0.3333 \text{ A}
\]

Hence, by (9-27), we obtain the sensitivity values

\[
\frac{\partial v_0}{\partial R_1} = -j_1 \hat{j}_1 \approx -(0.01333)(0.6667) \approx -0.00089 \text{ V/}\Omega
\]

and

\[
\frac{\partial v_0}{\partial R_2} = -j_2 \hat{j}_2 \approx (0.01333)(0.3333) \approx 0.00444 \text{ V/}\Omega
\]

which agree with the results obtained by differentiation in Sec. 9.1.

The process followed in this example will be generalized for an arbitrary linear resistive circuit. The circuit may also contain current-controlled voltage sources (CCVS). The branch relations of the resistors are

\[
v_k = R_k j_k \quad (9-28)
\]

and the controlled branch voltages of the CCVS satisfy

\[
v_i = R_{lm} j_m \quad (9-29)
\]

These branch relations can now be merged into the vector relation

\[
v_B = R j_B \quad (9-30)
\]

Here, \( R \) is the branch resistance matrix. The vectors \( v_B \) and \( j_B \) contain the variables for all internal branches of \( N \), that is, for those branches which do not contain independent sources. (These are contained inside the box \( N \) in Fig. 9-4a.) Similarly, let the source voltages and currents of \( N \) be collected in the vectors \( e \) and \( i \), respectively. A zero-valued current source is to be placed at an open-circuited output port (Fig. 9-6a); this source must be included in \( i \). Similarly, if the sensitivities \( \partial i_0 / \partial R_k \) of an output current \( i_0 \) are required, a zero-valued voltage source should be inserted in the output branch (Fig. 9-6b); it must be included in \( e \).

The branch variables of \( \hat{N} \) will similarly be collected in \( \hat{v}_B \) and \( \hat{j}_B \), the sources of \( \hat{N} \) in \( \hat{e} \) and \( \hat{i} \). Then (9-16) can be written in the form

\[
(\hat{j}_e^T \Delta e - \hat{e}_e^T \Delta j_e) + (\hat{j}_B^T \Delta v_B - \hat{v}_B^T \Delta j_B) + (\hat{i}_i^T \Delta v_i - \hat{v}_i^T \Delta i) = 0 \quad (9-31)
\]

Since we assume constant sources, \( \Delta e = 0 \) and \( \Delta i = 0 \). Also, from (9-30),

\[
\Delta v_B = \Delta R j_B + R \Delta j_B \quad (9-32)
\]

where we neglected the second-order small term \( \Delta R \Delta j_B \) just as we did in (9-20). Then

\[
-\hat{e}_e^T \Delta j_e + \hat{i}_i^T \Delta v_i = -\hat{j}_B^T (\Delta R j_B + R \Delta j_B) + \hat{v}_B^T \Delta j_B = -\hat{j}_B^T \Delta R j_B + (\hat{v}_B - \hat{j}_B^T R) \Delta j_B
\]

\[
(9-33)
\]
The terms $\Delta R_k$, $\Delta R_{in}$ are contained in $\Delta \mathbf{R}$; to eliminate all other terms except $-J_B^T \Delta \mathbf{R} J_B$ on the right-hand side we set

$$\hat{\mathbf{v}}_B^T - J_B^T \mathbf{R} = 0$$  \hspace{1cm} (9-34)

or

$$\hat{\mathbf{v}}_B = \mathbf{R} J_B^T$$  \hspace{1cm} (9-35)

Equation (9-35) gives the internal branch relations of $\hat{N}$. A comparison with (9-30) shows that the branch relations for $N$ and $\hat{N}$ are different in that $\mathbf{R}$ has been replaced by $\hat{\mathbf{R}} \triangleq \mathbf{R}^T$ for $\hat{N}$. However, if $N$ contains only resistors and no CCVS, all branch relations are of the form of (9-28) and hence $\mathbf{R}$ is diagonal. Then $\mathbf{R}^T = \mathbf{R}$, and the branch relations for $N$ and $\hat{N}$ are the same.

**Example 9-2** For the simple circuit of Fig. 9-5 the branch relations were

$$v_1 = R_1 j_1 \quad v_2 = R_2 j_2$$  \hspace{1cm} (9-36)

so that in vector form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$$  \hspace{1cm} (9-37)

with $\mathbf{R}$ a diagonal matrix. For this network therefore, $\mathbf{R}^T = \mathbf{R}$. This implies that the internal branches of $N$ and $\hat{N}$ are identical, as shown in Fig. 9-5.

**Example 9-3** Consider, by contrast, the circuit of Fig. 9-7a. The internal branch relations are

$$v_1 = R_1 j_1 \quad v_2 = 0 \quad v_3 = R_3 j_2 \quad v_4 = R_4 j_4$$  \hspace{1cm} (9-38)

$$v_3 = r_{31} j_1 + r_{32} j_2 + r_{33} j_3 + r_{34} j_4$$
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Thus, the branch resistance matrix of $N$ is

$$ R = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} \quad (9-39) $$

while that of $\hat{N}$ should be chosen as

$$ \hat{R} = R^* = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} \quad (9-40) $$

$\hat{R}$ corresponds to the branch relations

$$ \hat{v}_1 = R_1 \hat{j}_1 \quad \hat{v}_2 = R_2 \hat{j}_2 \quad \hat{v}_3 = 0 \quad \hat{v}_4 = R_4 \hat{j}_4 $$

Hence, $\hat{N}$ has the structure shown in Fig. 9-7b. Its internal branches are different from those of $N$. Specifically, the controlling and controlled branches of the CCVS are interchanged in $\hat{N}$.

This result is easily generalized. Let the $m$th branch current control the $l$th branch voltage in a general circuit (Fig. 9-8a). Then the branch relations of the CCVS are

$$ v_i = R_{lm} j_m \quad v_m = 0 \quad (9-41) $$

**Figure 9-7** (a) Circuit containing a CCVS; (b) its adjoint network. The element values are $e_1 = 5 \text{ V}, R_1 = 10 \text{ k}\Omega, R_4 = 50 \text{ k}\Omega, \text{ and } R_{32} = 20 \text{ k}\Omega.$
so that the branch resistance matrix $\mathbf{R}$ of $N$ is

$$
\mathbf{R} = \begin{bmatrix}
0 & 0 & \cdots & 0 & R_{lm} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

(9-42)

The branch resistance matrix $\hat{\mathbf{R}}$ of $\hat{N}$ is then the transpose of $\mathbf{R}$, or

$$
\hat{\mathbf{R}} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & R_{lm} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

(9-43)

Evidently now the current of branch $l$ controls the voltage of branch $m$; that is, the controlling and controlled branches have changed places† (Fig. 9-8b).

† Note that the controlling branch of the CCVS in the above derivations contains no circuit elements; it is a short circuit. This definition of the branch is for convenience; it simplifies the discussions somewhat.
Let us return now to Eq. (9-33). If (9-34) holds, then the right-hand side of (9-33) becomes

\[-\mathbf{j}_B^T \Delta \mathbf{R} \mathbf{j}_B = - \sum_{i, m} \hat{j}_i \hat{j}_m \Delta R_{im}\]

\[
\Delta v_v = (\Delta R_i + \ldots 
\Delta j_A = (\Delta R_i + \ldots 

If the output is the voltage \(v_{ik}\), choosing all \(\hat{e}_i = 0\) and all \(\hat{i}_i = 0\) except for \(\hat{i}_k = 1\) A causes the left-hand side of (9-33) to be equal to \(\Delta v_{ik}\). If the output is the current \(j_{ek}\), then choosing all \(\hat{i}_i = 0\) and all \(\hat{e}_i = 0\) except for \(\hat{e}_k = -1\) V makes the left-hand side equal to \(\Delta j_{ek}\). Hence, (9-33) becomes

\[
\Delta v_{ik} \text{ or } \Delta j_{ek} = - \sum_{i \neq m} \hat{j}_i \hat{j}_m \Delta R_{im} - \sum_{i \neq m} \hat{j}_i \hat{j}_m \Delta R_{im} \tag{9-45}
\]

Thus the sensitivity of the output to a resistance \(R_i\) is \(-\hat{j}_i\hat{j}_i\); to the gain \(R_{im}\) of a CCVS (Fig. 9-8a) it is \(-\hat{j}_i\hat{j}_m\). Note that the controlling currents \(\hat{j}_i\) and \(\hat{j}_m\) enter the sensitivity expression for the CCVS.

We can now recapitulate the steps leading to the sensitivities of a resistive circuit containing CCVSs:

1. Construct the internal branches of \(\hat{N}\) from those of \(N\). Specifically, branches containing resistors in \(N\) are simply duplicated in \(\hat{N}\). Branches containing the controlling and controlled variables of a CCVS in \(N\) are interchanged in \(\hat{N}\).
2. Set all independent sources to zero in \(\hat{N}\) except the one at the output port. If the output quantity is a voltage, excite the output port with a 1-A current source; if it is a current, excite the output port with a \(-1\)-V voltage source.
3. Analyze \(N\) and \(\hat{N}\) to find all internal branch currents \(j_k\) and \(\hat{j}_i\).
4. The sensitivities of the output to resistances are given by

\[
\frac{\partial v_{ik}}{\partial R_i} \text{ or } \frac{\partial j_{ek}}{\partial R_i} = -\hat{j}_i j_i \tag{9-46}
\]

and to CCVS gains by

\[
\frac{\partial v_{ik}}{\partial R_{im}} \text{ or } \frac{\partial j_{ek}}{\partial R_{im}} = -\hat{j}_i j_m \tag{9-47}
\]

**Example 9-4** For the circuit of Fig. 9-7a, by inspection \(j_1 = j_2 = e_1/R_1 = 0.5\) mA, \(j_3 = j_4 = -R_{32}j_2/R_4 = -0.2\) mA. In the adjoint network (Fig. 9-7b), we set \(\hat{e}_1 = 0\) and \(\hat{e}_2 = -1\) V. Then \(\hat{j}_3 = -\hat{j}_4 = \hat{e}_2/R_4 = -0.02\) mA, and \(\hat{j}_1 = \hat{j}_2 = -R_{32}j_3/R_1 = 0.04\) mA. Hence, the sensitivities are

\[
\frac{\partial j_0}{\partial R_1} = -\hat{j}_1 j_1 = -0.02 \times 10^{-6} \text{ A/Ω} = -0.02 \text{ mA/kΩ}
\]

\[
\frac{\partial j_0}{\partial R_4} = -\hat{j}_4 j_4 = -0.004 \times 10^{-6} \text{ A/Ω} = -0.004 \text{ mA/kΩ}
\]

\[
\frac{\partial j_0}{\partial R_{32}} = -\hat{j}_3 j_2 = 0.01 \times 10^{-6} \text{ A/Ω} = 0.01 \text{ mA/kΩ}
\]
As a test, we can perform a mathematical analysis of the circuit. This gives for the output current

\[ j_0 = \frac{e_1 R_{32}}{R_1 R_4} \]

and hence the sensitivities

\[ \frac{\partial j_0}{\partial R_1} = -\frac{e_1 R_{32}}{R_1^2 R_4} = -0.02 \text{ mA/k}\Omega \]

\[ \frac{\partial j_0}{\partial R_4} = -\frac{e_1 R_{32}}{R_1 R_4^2} = -0.004 \text{ mA/k}\Omega \]

\[ \frac{\partial j_0}{\partial R_{32}} = \frac{e_1}{R_1 R_4} = 0.01 \text{ mA/k}\Omega \]

which agree with the results obtained using the adjoint-network approach.

Consider now a circuit containing in its internal branches only resistors and voltage-controlled current sources (VCCS). Such circuits will have the internal branch relations \( j_k = (1/R_k)v_k = G_k v_k \) for the resistors and \( j_i = G_{lm} v_m \) for the VCCS.

These relations can be assembled into the vector relation

\[ \hat{j}_B = Gv_B \] (9-48)

where \( G \) is the branch conductance matrix.

An argument dual to that proving (9-45) now shows that if \( G^T \) is chosen as the branch conductance matrix \( \hat{G} \) of \( \hat{N} \), then (9-31) becomes

\[ -\hat{e}^T \Delta \hat{j}_c + \hat{i}^T \Delta v_i = \hat{v}_B^T \Delta Gv_B = \sum_{\text{all branches}} \hat{v}_l v_m \Delta G_{lm} \]

\[ = \sum_{\text{all resistors}} \hat{v}_l v_l \Delta G_l + \sum_{\text{all VCCS} \ l \neq m} \hat{v}_l v_m \Delta G_{lm} \] (9-49)

The relation \( \hat{G} = G^T \) can again be shown to result in an interchange of the controlling and controlled branches of all VCCS (Fig. 9-9). If the \( \hat{e} \) and \( \hat{i} \) are chosen the same way as for the CCVS circuits, then (9-49) gives

\[ \Delta v_{ik} \text{ or } \Delta j_{ek} = \sum_{\text{all } G_l} \hat{v}_l v_l \Delta G_l + \sum_{\text{all VCCS} \ l \neq m} \hat{v}_l v_m \Delta G_{lm} \] (9-50)

Hence, the sensitivities of the output to the resistances are\(^\dagger\)

\[ \frac{\partial v_{ik}}{\partial G_l} \text{ or } \frac{\partial j_{ek}}{\partial G_l} = \hat{v}_l v_l \] (9-51)

\(^\dagger\) Note that (9-51) is equivalent to (9-46) since, say

\[ \frac{\partial v_{ik}}{\partial G_l} = \frac{\partial v_{ik}}{\partial (1/R_l)} \frac{\partial (1/R_l)}{\partial R_l} = -\frac{\hat{j}_l}{R_l} = -\frac{\hat{j}_l}{R_l^2} = \hat{v}_l v_l \]
and to VCCS gains

\[ \frac{\partial v_k}{\partial G_{lm}} \text{ or } \frac{\partial j_k}{\partial G_{lm}} = \hat{v}_l v_m \]  

(9-52)

Note that again only the controlling voltages enter the sensitivities.

**Example 9-5** Calculate the sensitivities of \( v_0 \) in the circuit of Fig. 9-10a.

The adjoint network \( \hat{N} \) is found by turning the VCCS around and choosing the generators \( \hat{i}_1 = 0 \) and \( \hat{i}_2 = 1 \text{ A} \) (Fig. 9-10b). Both \( N \) and \( \hat{N} \) can again be analyzed by inspection: from Fig. 9-10a, \( v_1 = v_2 = -i_1/G_1 = -5 \text{ V}, \ v_3 = v_4 = -G_{32} v_2/G_4 = 0.625 \text{ V} \). Similarly from Fig. 9-10b, with \( \hat{i}_1 = 0 \) and \( \hat{i}_2 = 1 \text{ A}, \hat{v}_3 = \hat{v}_4 = -\hat{i}_2/G_4 = -250 \text{ V} \) and \( \hat{v}_1 = \hat{v}_2 = -G_{32} \hat{i}_2/G_1 = 125 \text{ V} \).

Hence, by (9-51) and (9-52), the sensitivities are

\[ \frac{\partial v_0}{\partial G_1} = \hat{v}_1 v_1 = -625 \text{ V/\Omega} = -0.625 \text{ V/m\Omega} \]

\[ \frac{\partial v_0}{\partial G_4} = \hat{v}_4 v_4 = -156.25 \text{ V/\Omega} \approx -0.156 \text{ V/m\Omega} \]

\[ \frac{\partial v_0}{\partial G_{32}} = \hat{v}_3 v_2 = 1250 \text{ V/\Omega} = 1.25 \text{ V/m\Omega} \]

† The large \( \hat{v} \) values are due to the choice of \( \hat{i}_2 = 1 \text{ A}, \) a convenient but uncommonly large value.
Mathematical analysis, performed as a test, gives the output voltage

\[ v_0 = \frac{i_1 G_{32}}{G_1 G_4} \]

and hence

\[ \frac{\delta v_0}{\delta G_1} = -\frac{i_1 G_{32}}{G_1^2 G_4} = -0.625 \text{ V/m} \Omega \]

\[ \frac{\delta v_0}{\delta G_4} = -\frac{i_1 G_{32}}{G_1 G_4^2} \approx -0.156 \text{ V/m} \Omega \]

\[ \frac{\delta v_0}{\delta G_{23}} = \frac{i_1}{G_1 G_4} = 1.25 \text{ V/m} \Omega \]

as before.

**Example 9-6** The previous examples were chosen to be simple, to permit easy understanding of the method and also easy checking. The following example† demonstrates the power of the method under practical circumstances. Consider the differential amplifier shown in Fig. 9-11a. We want to analyze the sensitivities of the small-signal voltage gain \( A_V = v_0/i_{in} \). If we assume the small-signal model shown in Fig. 9-11b for all transistors and utilize the symmetry of the circuit about ground (Fig. 9-11a) to bisect it, we get the half-circuit model of Fig. 9-11c. Computer analysis gives the branch currents indicated (in milliamperes) in this circuit. This analysis is routine, and many computer programs are available to carry it out.

Next, using the simple rules established earlier, the adjoint network of the circuit of Fig. 9-11c is drawn. It is shown in Fig. 9-11d. The branch current values (in amperes) are also indicated and are also obtained routinely on the computer.

From the voltages and currents shown in Fig. 9-11c and d, all sensitivities of \( v_0/2 \) can immediately be found using (9-51) [or its equivalent, (9-46)] and (9-52). For example, the sensitivity of \( v_0/2 \) to the 100-\( \Omega \) resistor is

\[
\frac{\delta(v_0/2)}{\delta R} = -jR_2 R_2 = (-9.4)(4.9 \times 10^{-3}) \approx -0.046 \text{ V/}\Omega
\]

while the sensitivity to the transconductance of the first transistor is

\[
\frac{\delta(v_0/2)}{\delta G_{m_1}} = \hat{I}_{c_1} v_{b_1} = (306)(6.46 \times 10^{-3}) \approx 1.975 \text{ V/}\mu\text{A} \approx 2 \text{ mV/m}\Omega
\]

It should be evident that for this circuit the analytic approach, consisting of finding an explicit mathematical expression for \( v_0/2 = f(R_1, R_2, \ldots, G_{m_1}, G_{m_2}) \)

Hybrid equations:

(general):

\[
x = \begin{bmatrix} V_{B1} \\ j_{B1} \end{bmatrix}, \quad y = \begin{bmatrix} j_{B2} \\ V_{B2} \end{bmatrix}
\]

VCVS, CCCS don't fit into \( R \) or \( G \):

Hybrid matrix:

\[
\mathbf{H} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \mathbf{H}^T \mathbf{H} = \begin{bmatrix} G & A \\ A^T & \mathbf{P} \end{bmatrix}
\]
and differentiating it formally, is impractical. Yet this is a very small circuit compared with those used routinely in, say, integrated op-amps. For such large circuits the computer analysis of the branch voltages and currents is still practical and hence so is the adjoint network technique of sensitivity analysis. Other approaches (mathematical differentiation, difference quotients) are impossible or at least prohibitively expensive.
Table 9-1  Circuit elements together with their adjoint counterparts and sensitivities

<table>
<thead>
<tr>
<th>Element Type</th>
<th>Branch Relation</th>
<th>Symbol</th>
<th>Element Type</th>
<th>Branch Relation</th>
<th>Symbol</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_i$</td>
<td>$v_i = R_i j_i$</td>
<td>![Ri Symbol]</td>
<td>$R_i$</td>
<td>$\tilde{v}_i = R_i \tilde{j}_i$</td>
<td>![Ri Symbol]</td>
<td>$S_{R_i} = \frac{\partial v_o}{\partial R_i}$ or $\frac{\partial j_o}{\partial R_i}$</td>
</tr>
<tr>
<td>$G_i$</td>
<td>$j_i = G_i v_i$</td>
<td>![Gi Symbol]</td>
<td>$G_i$</td>
<td>$\tilde{j}_i = G_i \tilde{v}_i$</td>
<td>![Gi Symbol]</td>
<td>$S_{G_i} = \tilde{v}_i v_i$</td>
</tr>
<tr>
<td>$CCVS$</td>
<td>$v_m = 0$</td>
<td>$j_m$</td>
<td>$CCVS$</td>
<td>$\tilde{v}<em>m = R</em>{im} \tilde{j}_i$</td>
<td>![Ccvs Symbol]</td>
<td>$S_{R_{im}} = -\tilde{j}_i j_m$</td>
</tr>
<tr>
<td>$VCCS$</td>
<td>$j_m = 0$</td>
<td>$\tilde{v}_m$</td>
<td>$VCCS$</td>
<td>$\tilde{j}<em>i = G</em>{im} \tilde{v}_i$</td>
<td>![Vccs Symbol]</td>
<td>$S_{G_{im}} = \tilde{v}_i v_m$</td>
</tr>
<tr>
<td>$VCVS$</td>
<td>$j_m = 0$</td>
<td>$v_m$</td>
<td>$VCVS$</td>
<td>$\tilde{j}<em>i = M</em>{im} \tilde{v}_i$</td>
<td>![Vcvs Symbol]</td>
<td>$S_{M_{im}} = -\tilde{j}_i v_m$</td>
</tr>
<tr>
<td>$CCCS$</td>
<td>$v_m = 0$</td>
<td>$j_m$</td>
<td>$CCCS$</td>
<td>$\tilde{v}<em>m = -A</em>{im} \tilde{v}_i$</td>
<td>![Ccvs Symbol]</td>
<td>$S_{A_{im}} = \tilde{v}_i j_m$</td>
</tr>
<tr>
<td>Excitations of $\tilde{N}$: 1A for $\frac{\partial v_o}{\partial x}, \Delta v_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1A for $\frac{\partial j_o}{\partial x}, \Delta j_0$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>