Two-port networks are constrained to have the same currents at each port: \( I_1' = I_1, \ I_2' = I_2 \). Hence, there are 4 unknowns: \( I_1, \ V_1, \ I_2 \) and \( V_2 \). The source and load provide 2 relations between these, so the two-port must give 2 more for an analyzable linear circuit.

**Figure 5-1** Two-port network.

**Figure 5-3** Circuits whose structure assures the satisfaction of the two-port conditions. (a) The internal structure which forces (5-1) to hold. (b) The transformers at the two ports.
3.3 TWO-PORT NETWORKS

At this point it is possible to proceed by treating the general multiport network and discussing sets of equations relating the port variables. After this is done, the results can be applied to the special case of a two-port. An alternative approach is to treat the simplest multiport (namely, the two-port) first. This might be done because of the importance of the two-port in its own right, and because treating the simplest case first can lead to insights into the general case that will not be obvious without experience with the simplest case. We shall take this second approach.

A two-port network is illustrated in Fig. 7. Because of the application of two-ports as transmission networks, one of the ports—normally the port labeled 1—is called the *input*; the other, port 2, is called the *output*. The port variables are two port currents and two port voltages, with the standard references shown in Fig. 7. (In some of the literature the reference for $I_2$ is taken opposite to our reference. When comparing any formulas in other publications, verify the references of the port parameters.) External networks that may be connected at the input and output are called the *terminations*. We shall deal throughout with the transformed variables and shall assume the two-port to be initially relaxed and to contain no independent sources.

The discussion that follows may appear somewhat unmotivated, since in restricting ourselves to analysis we have lost much of the motivation for finding various ways of describing the behavior of two-port networks. The need for these various schemes arises from the demands made by the many applications of two-ports. The usefulness of the different methods of description comes clearly into evidence when the problem is one of synthesizing or designing networks—filters, matching networks, wave-shaping networks, and a host of others. A method of description that is convenient for a power system, say, may be less so for a filter network, and may be completely unsuited for a transistor amplifier. For
this reason we shall describe many alternative, but equivalent, ways of describing two-port behavior.

In the problem of synthesizing a network for a specific application, it is often very convenient to break down a complicated problem into several parts. The pieces of the overall network are designed separately and then put together in a manner consistent with the original decomposition. In order to carry out this procedure it is necessary to know how the description of the behavior of the overall network is related to the behavior of the components. For this reason we shall spend some time on the problem of interconnecting two-ports.

Many of the results obtained in this section require a considerable amount of algebraic manipulation that is quite straightforward. We shall not attempt to carry through all the steps, but shall merely outline the desired procedure, leaving to you the task of filling in the omitted steps.

OPEN- AND SHORT-CIRCUIT PARAMETERS

To describe the relationships among the port voltages and currents of a linear multiport requires as many linear equations as there are ports. Thus for a two-port two linear equations are required among the four variables. However, which two variables are considered "independent" and which "dependent" is a matter of choice and convenience in a given application. To return briefly to the general case, in an $n$-port, there will be $2n$ voltage-and-current variables. The number of ways in which these $2n$ variables can be arranged in two groups of $n$ each equals the number of ways in which $2n$ things can be taken $n$ at a time; namely, $(2n)!/(n!)^2$. For a two-port this number is 6.

One set of equations results when the two-port currents are expressed in terms of the two-port voltages:

$$
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix} =
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
V_1(s) \\
V_2(s)
\end{bmatrix},
$$

(10)

It is a simple matter to obtain interpretations for these parameters by letting each of the voltages be zero in turn. It follows from the equation that

$$
y_{11}(s) = \frac{I_1(s)}{V_1(s)} \bigg|_{v_2=0}, \quad y_{12}(s) = \frac{I_1(s)}{V_2(s)} \bigg|_{v_1=0},
$$

$$
y_{21}(s) = \frac{I_2(s)}{V_1(s)} \bigg|_{v_2=0}, \quad y_{22}(s) = \frac{I_2(s)}{V_2(s)} \bigg|_{v_1=0}.
$$

(11)
Dimensionally, each parameter is an admittance. Setting a port voltage to zero means short-circuiting the port. Hence the $y$’s (for which the lower case letter $y$ will be reserved) are called the short-circuit admittance parameters (the $y$-parameters for short). The matrix of $y$’s is designated $Y_{sc}$ and is called the short-circuit admittance matrix. The terms $y_{11}$ and $y_{22}$ are the short-circuit driving-point admittances at the two ports, and $y_{21}$ and $y_{12}$ are short-circuit transfer admittances. In particular, $y_{21}$ is the forward transfer admittance—that is, the ratio of a current response in port 2 to a voltage excitation in port 1—and $y_{12}$ is the reverse transfer admittance.

A second set of relationships can be written by expressing the port voltages in terms of the port currents:

$$
\begin{bmatrix}
V_1(s) \\
V_2(s)
\end{bmatrix} =
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix}.
$$

(12)

This time interpretations are obtained by letting each current be zero in turn. Then

$$
\begin{align*}
z_{11}(s) &= \left. \frac{V_1(s)}{I_1(s)} \right|_{I_2=0}, & z_{12}(s) &= \left. \frac{V_1(s)}{I_2(s)} \right|_{I_1=0}, \\
z_{21}(s) &= \left. \frac{V_2(s)}{I_1(s)} \right|_{I_2=0}, & z_{22}(s) &= \left. \frac{V_2(s)}{I_2(s)} \right|_{I_1=0}.
\end{align*}
$$

(13)

Dimensionally, each parameter is an impedance. Setting a port current equal to zero means open-circuiting the port. Hence the $z$’s (for which the lower case letter $z$ will be reserved) are called the open-circuit impedance parameters (the $z$-parameters for short). The matrix of $z$’s is designated $Z_{oc}$ and is called the open-circuit impedance matrix. The elements $z_{11}$ and $z_{22}$ are the driving-point impedances at the two ports, and $z_{21}$ and $z_{12}$ are the transfer impedances; $z_{21}$ is the forward transfer impedance, and $z_{12}$ is the reverse transfer impedance.

It should be clear from (10) and (12) that the $Y_{sc}$ and $Z_{oc}$ matrices are inverses of each other; for example,

$$
Y_{sc} = \begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix} = Z_{oc}^{-1} = \frac{1}{\det Z_{oc}} \begin{bmatrix}
z_{22} & -z_{12} \\
-z_{21} & z_{11}
\end{bmatrix}.
$$

(14)

From this it follows that

$$
\det Y_{sc} = \frac{1}{\det Z_{oc}}.
$$

(15)

Demonstration of this is left as an exercise.
The results developed so far apply whether the network is passive or active, reciprocal or nonreciprocal. Now consider the two transfer functions \( y_{21} \) and \( y_{12} \). If the network is reciprocal, according to the definition in Section 1.4, they will be equal. So also will \( z_{12} \) and \( z_{21} \); that is, for a reciprocal network

\[
y_{12} = y_{21}, \quad z_{12} = z_{21}, \tag{16}
\]

which means that both \( Y_{sc} \) and \( Z_{oc} \) are symmetrical for reciprocal networks.

**HYBRID PARAMETERS**

The \( z \) and \( y \) representations are two of the ways in which the relationships among the port variables can be expressed. They express the two voltages in terms of the two currents, and vice versa. Two other sets of equations can be obtained by expressing a current and voltage from opposite ports in terms of the other voltage and current. Thus

\[
\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \tag{17}
\]

and

\[
\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}. \tag{18}
\]

The interpretations of these parameters can be easily determined from the preceding equations to be the following:

\[
\begin{align*}
    h_{11} & = \left. \frac{V_1(s)}{I_1(s)} \right|_{I_2=0}, & h_{12} & = \left. \frac{V_1(s)}{V_2(s)} \right|_{I_1=0}, \\
    h_{21} & = \left. \frac{I_2(s)}{I_1(s)} \right|_{V_2=0}, & h_{22} & = \left. \frac{I_2(s)}{V_2(s)} \right|_{I_1=0}, \\
    g_{11} & = \left. \frac{I_1(s)}{V_1(s)} \right|_{I_2=0}, & g_{12} & = \left. \frac{I_1(s)}{I_2(s)} \right|_{V_1=0}, \\
    g_{21} & = \left. \frac{V_2(s)}{V_1(s)} \right|_{I_2=0}, & g_{22} & = \left. \frac{V_2(s)}{I_2(s)} \right|_{V_1=0}.
\end{align*} \tag{19}
\]
Thus we see that the $h$- and $g$-parameters are interpreted under a mixed set of terminal conditions, some of them under open-circuit and some under short-circuit conditions. They are called the hybrid $h$- and hybrid $g$-parameters. From these interpretations we see that $h_{11}$ and $g_{22}$ are impedances, whereas $h_{22}$ and $g_{11}$ are admittances. They are related to the $z$'s and $y$'s by

$$
\begin{align*}
    h_{11} &= \frac{1}{y_{11}}, & g_{11} &= \frac{1}{z_{11}}, \\
    h_{22} &= \frac{1}{z_{22}}, & g_{22} &= \frac{1}{y_{22}}.
\end{align*}
$$

(20)

The transfer $g$'s and $h$'s are dimensionless. The quantity $h_{21}$ is the forward short-circuit current gain, and $g_{12}$ is the reverse short-circuit current gain. The other two are voltage ratios; $g_{21}$ is the forward open-circuit voltage gain, whereas $h_{12}$ is the reverse open-circuit voltage gain. We shall use $H$ and $G$ to represent the corresponding matrices.

By direct computation we find the following relations among the transfer parameters:

$$
\begin{align*}
    h_{12} &= \frac{-z_{12}}{z_{21}} h_{21}, \\
    g_{12} &= \frac{-y_{12}}{y_{21}} g_{21}.
\end{align*}
$$

(21a)

(21b)

In the special case of reciprocal networks these expressions simplify to $h_{12} = -h_{21}$ and $g_{12} = -g_{21}$. In words this means that for reciprocal networks the open-circuit voltage gain for transmission in one direction through the two-port equals the negative of the short-circuit current gain for transmission in the opposite direction.

Just as $Z_{oc}$ and $Y_{sc}$ are each the other’s inverse, so also $H$ and $G$ are each the other’s inverse. Thus

$$
G(s) = H^{-1}(s), \quad \det G = \frac{1}{\det H}.
$$

(22)

You should verify this.
CHAIN PARAMETERS

The remaining two sets of equations relating the port variables express the voltage and current at one port in terms of the voltage and current at the other. These were, in fact, historically the first set used—in the analysis of transmission lines. One of these equations is

\[
\begin{bmatrix}
V_1(s) \\
I_1(s)
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(s) \\
-I_2(s)
\end{bmatrix}.
\]

They are called the chain, or ABCD, parameters. The first name comes from the fact that they are the natural ones to use in a cascade, or tandem, or chain connection typical of a transmission system. Note the negative sign in \(-I_2\), which is a consequence of the choice of reference for \(I_2\).

Note that we are using the historical symbols for these parameters rather than using, say, \(a_{ij}\) for \(i\) and \(j\) equal 1 and 2, to make the system of notation uniform for all the parameters. We are also not introducing further notation to define the inverse parameters obtained by inverting (23), simply to avoid further proliferation of symbols.

The determinant of the chain matrix can be computed in terms of \(z\)'s and \(y\)'s. It is found to be

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = AD - BC = \frac{z_{12}}{z_{21}} = \frac{y_{12}}{y_{21}},
\]

which is equal to 1 for reciprocal two-ports.

The preceding discussion is rather detailed and can become tedious if one loses sight of the objective of developing methods of representing the external behavior of two-ports by giving various relationships among the port voltages and currents. Each of these sets of relationships finds useful applications. For future reference we shall tabulate the interrelationships among the various parameters. The result is given in Table 1. Note that these relationships are valid for a general nonreciprocal two-port.

TRANSMISSION ZEROS

There is an important observation that can be made concerning the locations of the zeros of the various transfer functions. This can be seen most readily, perhaps, by looking at one of the columns in Table 1; for
Example:

\[ V_1 = AV_2 - BI_2 \]
\[ I_1 = CV_2 - DI_2 \]

\[ V_2 = \frac{-R_2}{R_1} V_1 \quad \rightarrow \quad A = \frac{R_1}{R_2}, \quad B = 0 \]

\[ I_1 = \frac{V_1}{R_1} = A \frac{V_2}{R_1} = -\frac{V_2}{R_2} \quad \rightarrow \quad C = -\frac{1}{R_2}, \quad D = 0 \]

\[ AD - BC = 0 - 0 \neq 1 \]

\[
\begin{bmatrix}
\frac{-R_1}{R_2} & 0 \\
\frac{R_2}{R_1} & 0 \\
-\frac{1}{R_2} & 0
\end{bmatrix} \quad \Rightarrow \quad \text{non-reciprocal}
\]
A transmission zero is a value of $s$ (or frequency) for which the output signal equals zero even though the input was nonzero. Example: ladder network

\[ V_{out} = \frac{1}{s} V \]

Transmission zeros
\[ \pm j \omega_1, \pm j \omega_2 \]

No transmission zeros

**Example**

As an illustrative example of the computation of two-port parameters, consider the network shown in Fig. 8, which can be considered as a model for a vacuum triode under certain conditions. (The capacitances are the grid-to-plate and plate-to-cathode capacitances.) Let us compute the $y$-parameters for this network. The simplest procedure is to use the interpretations in (11). If the output terminals are short-circuited, the resulting network will take the form shown in Fig. 9. As far as the input terminals are concerned, the controlled source has no effect. Hence $y_{11}$ is the admittance of the parallel combination of $R_g$ and $C_1$:

\[ y_{11}(s) = \frac{1}{R_g} + sC_1. \]

Parallel branches across ports $\rightarrow y_{ij}$ simplest!
To find $y_{21}$, assume that a voltage source with transform $V_1(s)$ is applied at the input terminals. By applying Kirchhoff's current law at the node labeled 1 in Fig. 9, we find that $I_2 = g_m V_1 - sC_1 V_1$. Hence $y_{21}$ becomes

$$y_{21} = \frac{I_2(s)}{V_1(s)} \bigg|_{V_2=0} = g_m - sC_1.$$ 

Now short-circuit the input terminals of the original network. The result will take the form in Fig. 10. Since $V_1$ is zero, the dependent source current is also zero. It is now a simple matter to compute $y_{22}$ and $y_{12}$:

$$y_{22} = \frac{I_2}{V_2} \bigg|_{V_1=0} = s(C_1 + C_2) + \frac{1}{R_p},$$

$$y_{12} = \frac{I_1}{V_2} \bigg|_{V_1=0} = -sC_1.$$ 

We see that $y_{12}$ is different from $y_{21}$, as it should be, because of the presence of the controlled source.

If the $y$-parameters are known, any of the other sets of parameters can be computed by using Table 1. Note that even under the conditions that $C_1$ and $C_2$ are zero and $R_g$ infinite, the $y$-parameters exist, but the $z$-parameters do not ($z_{11}$, $z_{22}$, and $z_{21}$ become infinite).
How to find a new set of two-port parameters from a given one:

Example: Find the $H$ parameters from the $Z$ ones. We want

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

From second $Z$ equation:

$$V_2 = z_{21}I_1 + z_{22}I_2$$
$$I_2 = \left( \frac{I}{h_{22}} \right) V_2 - \left( \frac{z_{21}}{h_{22}} \right) I_1$$

Plug $I_2$ in first equation:

$$V_1 = z_{11}I_1 + z_{12}I_2 = z_{11}I_1 + \left( \frac{z_{12}}{z_{22}} \right) (V_2 - z_{21}I_1)$$
$$V_1 = \frac{z_{12}/z_{22}}{h_{12}} V_2 + \frac{z_{11} - z_{12}z_{21}/z_{22}}{h_{11}}$$

Comparing (A) and (B) with the $H$ equations shows that $h_{11} = \det Z / z_{22}$, $h_{12} = z_{12} / z_{22}$, $h_{21} = -z_{21}/z_{22}$, and $h_{22} = 1/z_{22}$

See Table 1 on p. 166 in B&B for a complete set of formulas. In general, rewrite original Equations:

$$V_1 = z_{11}I_1 + z_{12}I_2 \rightarrow V_1 - z_{12}I_2 = (z_{11}I_1) = a_1$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \rightarrow 0 + z_{22}I_2 = (v_2 - z_{21}I_1) = a_2$$

Solve for $V_1$ and $I_2$ in terms of $I_1, V_2$. 
<table>
<thead>
<tr>
<th>Chain Parameters</th>
<th>Hybrid g-Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$AD - BC$</td>
</tr>
<tr>
<td>$B$</td>
<td>$D - (AD - BC)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Short-Circuit Admittance Parameters</th>
<th>Hybrid h-Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_{11}$</td>
<td>$Y_{12}$</td>
</tr>
<tr>
<td>$Y_{21}$</td>
<td>$Y_{22}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Open-Circuit Impedance Parameters</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_{11}$</td>
<td>$Z_{12}$</td>
</tr>
<tr>
<td>$Z_{21}$</td>
<td>$Z_{22}$</td>
</tr>
</tbody>
</table>

Table 1
3.4 INTERCONNECTION OF TWO-PORT NETWORKS

A given two-port network having some degree of complexity can be viewed as being constructed from simpler two-port networks whose ports are interconnected in certain ways. Conversely, a two-port network that is to be built can be designed by combining simple two-port structures as building blocks. From the designer’s standpoint it is much easier to design simple blocks and to interconnect them than to design a complex network in one piece. A further practical reason for this approach is that it is much easier to shield smaller units and thus reduce parasitic capacitances to ground.

CASCADE CONNECTION

There are a number of ways in which two-ports can be interconnected. In the simplest interconnection of 2 two-ports, called the cascade, or tandem-connection, one port of each network is involved. Two two-ports are said to be connected in cascade if the output port of one is the input port of the second, as shown in Fig. 11.

![Figure 11. Cascade connection of two-ports.](image)

Our interest in the problem of "interconnection" is, from the analysis point of view, to study how the parameters of the overall network are related to the parameters of the individual building blocks. The tandem combination is most conveniently studied by means of the \(ABCD\)-parameters. From the references in the figure we see that

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix}, \quad \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix}, \quad \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} = \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
\]

Hence for the \(ABCD\) system of equations of the network \(N_b\) we can write

\[
\begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
\]
Furthermore, if we write the $ABCD$ system of equations for the network $N_a$ and substitute in the last equation, we get

$$
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A_a & B_a \\
C_a & D_a
\end{bmatrix}
\begin{bmatrix}
V_{2a} \\
-I_{2a}
\end{bmatrix} =
\begin{bmatrix}
A_b & B_b \\
C_b & D_b
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
$$

Thus the $ABCD$-matrix of two-ports in cascade is equal to the product of the $ABCD$-matrices of the individual networks; that is,

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
A_a & B_a \\
C_a & D_a
\end{bmatrix}
\begin{bmatrix}
A_b & B_b \\
C_b & D_b
\end{bmatrix}.
$$

Once the relationships between the parameters of the overall two-port and those of the components are known for any one set of parameters, it is merely algebraic computation to get the relationships for any other set; for example, the open-circuit parameters of the overall two-port can be found in terms of those for each of the two cascaded ones by expressing the $z$-parameters in terms of the $ABCD$-parameters for the overall network, using (26) and then expressing the $ABCD$-parameters for each network in the cascade in terms of their corresponding $z$-parameters. The result will be

$$
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix} =
\begin{bmatrix}
z_{11a} - \frac{z_{12a}z_{21a}}{z_{22a} + z_{11b}} & \frac{z_{12a}z_{12b}}{z_{22a} + z_{11b}} \\
\frac{z_{21a}z_{21b}}{z_{22a} + z_{11b}} & z_{22b} - \frac{z_{12b}z_{21b}}{z_{22a} + z_{11b}}
\end{bmatrix}.
$$

The details of this computation are left to you.

A word of caution is necessary. When it is desired to determine some specific parameter of an overall two-port in terms of parameters of the components in the interconnection, it may be simpler to use a direct analysis than to rely on relationships such as those in Table 1. As an example, suppose it is desired to find the expression for $z_{21}$ in Fig. 11. The term $z_{21}$ is the ratio of open-circuit output voltage to input current: $z_{21} = V_2/I_1$. Suppose a current source $I_1$ is applied; looking into the output terminals of $N_a$, let the network be replaced by its Thévenin equivalent. The result is shown in Fig. 12. By definition, $z_{21b} = V_2/I_{1b}$
Fig. 12. Replacement of network $N_a$ by its Thévenin equivalent.

with the output terminals open. Now $I_{1b}$ can easily be found from the network in Fig. 12 to be

$$I_{1b} = \frac{z_{21a} I_1}{z_{22a} + z_{11b}}.$$  

Hence

$$z_{21b} = \frac{V_2}{I_{1b}} = \frac{V_2}{z_{21a} I_1} = \frac{(z_{22a} + z_{11b}) V_2}{z_{21a} I_1}.$$  

Finally,

$$z_{21} = \frac{V_2}{I_1} = \frac{z_{21a} z_{21b}}{z_{22a} + z_{11b}},$$  \hspace{1cm} (28)

which agrees with (27).

An important feature of cascaded two-ports is observed from the expressions for the transfer impedances in (27). The zeros of $z_{21}$ are the zeros of $z_{21a}$ and $z_{21b}$. (A similar relationship holds for $z_{12}$.) Thus the transmission zeros of the overall cascade consist of the transmission zeros of each of the component two-ports. This is the basis of some important methods of network synthesis. It permits individual two-ports to be designed to achieve certain transmission zeros before they are connected together. It also permits independent adjustment and tuning of elements within each two-port to achieve the desired null without influencing the adjustment of the cascaded two-ports.

PARALLEL AND SERIES CONNECTIONS

Now let us turn to other interconnections of two-ports, which, unlike the cascade connection, involve both ports. Two possibilities that immediately come to mind are parallel and series connections. Two two-ports
are said to be connected in parallel if corresponding (input and output) ports are connected in parallel as in Fig. 13a. In the parallel connection

![Parallel connection diagram](image)

**Fig. 13.** Parallel and series connections of two-ports.

the input and output voltages of the component two-ports are forced to be the same, whereas the overall port currents equal the sums of the corresponding component port currents. This statement assumes that the port relationships of the individual two-ports are not altered when the connection is made. In this case the overall port relationship can be written as

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \begin{bmatrix} I_{1a} \\
I_{2a}
\end{bmatrix} + \begin{bmatrix} I_{1b} \\
I_{2b}
\end{bmatrix} = \begin{bmatrix} y_{11a} & y_{12a} \\
y_{21a} & y_{22a}
\end{bmatrix} \begin{bmatrix} V_{1a} \\
V_{2a}
\end{bmatrix} + \begin{bmatrix} y_{11b} & y_{12b} \\
y_{21b} & y_{22b}
\end{bmatrix} \begin{bmatrix} V_{1b} \\
V_{2b}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
y_{11a} + y_{11b} & y_{12a} + y_{12b} \\
y_{21a} + y_{21b} & y_{22a} + y_{22b}
\end{bmatrix} \begin{bmatrix} V_1 \\
V_2
\end{bmatrix}.
\]

That is, the short-circuit admittance matrix of two-ports connected in parallel equals the sum of the short-circuit admittance matrices of the component two-ports:

\[
Y_{sc} = Y_{sca} + Y_{scb}.
\tag{30}
\]

The dual of the parallel connection is the series connection. Two two-ports are connected in series if corresponding ports (input and output) are connected in series, as shown in Fig. 13b. In this connection the input and output port currents are forced to be the same, whereas the overall port voltages equal the sums of the corresponding port voltages of the individual two-ports. Again, it is assumed that the port relationships of the individual two-ports are not altered when the connection is made. In this case the overall port relationship can be written as
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
V_{1a} \\
V_{2a}
\end{bmatrix} + \begin{bmatrix}
V_{1b} \\
V_{2b}
\end{bmatrix} = \begin{bmatrix}
z_{11a} & z_{12a} \\
z_{21a} & z_{22a}
\end{bmatrix} \begin{bmatrix}
I_{1a} \\
I_{2a}
\end{bmatrix} + \begin{bmatrix}
z_{11b} & z_{12b} \\
z_{21b} & z_{22b}
\end{bmatrix} \begin{bmatrix}
I_{1b} \\
I_{2b}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
z_{11a} + z_{11b} & z_{12a} + z_{12b} \\
z_{21a} + z_{21b} & z_{22a} + z_{22b}
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\] (31)

That is, the open-circuit impedance matrix of two-ports connected in series equals the sum of the open-circuit impedance matrices of the component two-ports:

\[
Z_{oc} = Z_{oca} + Z_{ocb}.
\] (32)

Of these two—parallel and series connections—the parallel connection is more useful and finds wider application in synthesis. One reason for this is the practical one that permits two common-terminal (grounded) two-ports to be connected in parallel, the result being a common-terminal two-port. An example of this is the parallel-ladders network (of which the twin-tee null network is a special case) shown in Fig. 14.

---

![Parallel-ladders network diagram](image)

Fig. 14. Parallel-ladders network.

On the other hand, the series connection of two common-terminal two-ports is not a common-terminal two-port unless one of them is a tee network. Consider two grounded two-ports connected in series, as in Fig. 15a. It is clear that this is inadmissible, since the ground terminal of \(N_a\) will short out parts of \(N_b\), thus violating the condition that the individual two-ports be unaltered by the interconnection. The situation is remedied by making the common terminals of both two-ports common to each other, as in Fig. 15b. In this case the resulting two-port is not a common-terminal one. If one of the component two-ports is a tee, the series connection takes the form shown in Fig. 15c. This can be redrawn,
as in Fig. 15d, as a common-terminal two-port. That the last two networks have the same \( z \)-parameters is left for you to demonstrate.

\[ \text{Fig. 15. Series connection of common-terminal two-ports.} \]

Variations of the series and parallel types of interconnections are possible by connecting the ports in series at one end and in parallel at the other. These are referred to as the \textit{series-parallel} and \textit{parallel-series} connections. As one might surmise, it is the \( h \)- and \( g \)-parameters of the individual two-ports that are added to give the overall \( h \)- and \( g \)-parameters, respectively. This also is left as an exercise.

**PERMISSIBILITY OF INTERCONNECTION**

It remains for us to inquire into the conditions under which two-ports can be interconnected without causing the port relationships of the individual two-ports to be disturbed by the connection. For the parallel connection, consider Fig. 16. A pair of ports, one from each two-port, is

\[ \text{Fig. 16. Test for parallel-connected two-ports.} \]
Before added short, \( I_{1a} = I'_{1a} \) guaranteed; afterwards, if \( V=0 \), \( I_{1a} = 0 \), so \( I_{1a} = I'_{1a} \) still holds. Added short needed when finding \( y_{11} \) and \( y_{21} \).

\[
\begin{align*}
Z_{11} &= V_1 = Z_1 + V_{1a} + Z_3 = Z_{11a} + Z_{11b} \\
Z_{21} &= V_2 = V_{21} + Z_3 = Z_{21a} + Z_{21b}
\end{align*}
\]

Same result for \( I_2 = 1, I_1 = 0 \).
connected in parallel, whereas the other ports are individually short-circuited. The short circuits are employed because the parameters characterizing the individual two-ports and the overall two-port are the short-circuit admittance parameters. If the voltage $V$ shown in Fig. 16 is nonzero, then when the second ports are connected there will be a circulating current, as suggested in the diagram. Hence the condition that the current leaving one terminal of a port be equal to the current entering the other terminal of each individual two-port is violated, and the port relationships of the individual two-ports are altered.

For the case of the series connection, consider Fig. 17. A pair of ports,

![Diagram](image)

Fig. 17. Test for series-connected two-ports.

one from each two-port, is connected in series, whereas the other ports are left open. The open circuits are employed because the parameters characterizing the individual two-ports and the overall two-port are the open-circuit impedance parameters. If the voltage $V$ is nonzero, then when the second ports are connected in series there will be a circulating current, as suggested in the diagram. Again, the port relationships of the individual two-ports will be modified by the connection, and hence the addition of impedance parameters will not be valid for the overall network.

Obvious modifications of these tests apply to the series-parallel and parallel-series connections. The preceding discussion of the conditions under which the overall parameters for interconnected two-ports can be obtained by adding the component two-port parameters has been in rather skeletal form. We leave to you the task of supplying details.

When it is discovered that a particular interconnection cannot be made because circulating currents will be introduced, there is a way of stopping such currents and thus permitting the connection to be made. The approach is simply to put an isolating ideal transformer of 1:1 turns ratio at one of the ports, as illustrated in Fig. 18 for the case of the parallel connection.
3.5 MULTI PORT NETWORKS

The preceding section has dealt with two-port networks in considerable detail. Let us now turn our attention to networks having more than two ports. The ideas discussed in the last section apply also to multiports with obvious extensions.

Consider the $n$-port network shown in Fig. 19. The external behavior of this network is completely described by giving the relationships among the port voltages and currents. One such relationship expresses all the port voltages in terms of the port currents:

$$
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n
\end{bmatrix}
= 
\begin{bmatrix}
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
z_{n1} & z_{n2} & \cdots & z_{nn}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n
\end{bmatrix}
$$

(33a)
or

\[ V = Z_{oc} I. \]  \hspace{1cm} (33b)

By direct observation, it is seen that the parameters can be interpreted as

\[ z_{jk} = \frac{V_j}{I_k} \bigg| \text{all other currents} = 0 \]  \hspace{1cm} (34)

which is simply the extension of the open-circuit impedance representation of a two-port. The matrix \( Z_{oc} \) is the same as that in (12) except that it is of order \( n \).

The short-circuit admittance matrix for a two-port can also be directly extended to an \( n \)-port. Thus

\[ I = Y_{sc} V, \quad Y_{sc} = [y_{jk}] \]  \hspace{1cm} (35a)

where

\[ y_{jk} = \frac{I_j}{V_k} \bigg| \text{all other voltages} = 0 \]  \hspace{1cm} (35b)

If we now think of extending the hybrid representations of a two-port, we encounter some problems. In a hybrid representation the variables are mixed voltage and current. For a network of more than two ports, how are the "independent" and "dependent" variables to be chosen? In a three-port network, for example, the following three choices can be made:

\[
\begin{bmatrix}
V_1 \\
V_2 \\
I_3
\end{bmatrix} = M_1 \begin{bmatrix}
I_1 \\
I_2 \\
V_3
\end{bmatrix}, \quad \begin{bmatrix}
V_1 \\
I_2 \\
V_3
\end{bmatrix} = M_2 \begin{bmatrix}
I_1 \\
V_2 \\
I_3
\end{bmatrix}, \quad \begin{bmatrix}
I_1 \\
V_2 \\
V_3
\end{bmatrix} = M_3 \begin{bmatrix}
I_1 \\
V_2 \\
I_3
\end{bmatrix}
\]

as well as their inverses. In these choices each vector contains exactly one variable from each port. It would also be possible to make such selections as

\[
\begin{bmatrix}
V_1 \\
V_2 \\
I_2
\end{bmatrix} = M_4 \begin{bmatrix}
I_1 \\
I_3 \\
V_3
\end{bmatrix}
\]
Just as in the case of two-ports, it is possible to interconnect multi-ports. Two multi-ports are said to be connected in parallel if their ports are connected in parallel in pairs. It is not, in fact, necessary for the two multi-ports to have the same number of ports. The ports are connected in parallel in pairs until we run out of ports. It does not matter whether we run out for both networks at the same time or earlier for one network. Similarly, two multi-ports are said to be connected in series if their ports are connected in series of pairs. Again, the two multi-ports need not have the same number of ports.

As in the case of two-ports, the overall y-matrix for two n-ports connected in parallel equals the sum of the y-matrices of the individual n-ports. Similarly, the overall z-matrix of two n-ports connected in a series equals the sum of the z-matrices of the individual n-ports. This assumes, of course, that the interconnection does not alter the parameters of the individual n-ports.