Back to the chain matrix:

\[ V_1 = AV_2 - BI_2 \quad I_1 = CV_2 - DI_2 \]  \hfill (5-24)

\( A, B, C, \) and \( D \) are called the \textit{chain parameters}; their matrix

\[ T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]  \hfill (5-25)

\[ A = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{I_2=0} \quad B = \begin{pmatrix} V_1 \\ -I_2 \end{pmatrix}_{V_2=0} \quad C = \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}_{I_2=0} \quad D = \begin{pmatrix} I_1 \\ -I_2 \end{pmatrix}_{V_2=0} \]  \hfill (5-27)

Hence, \( T \) can be found from the schemes shown in Fig. 5-11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5-11}
\caption{Schemes for calculating the chain parameters of a two-port.}
\end{figure}
Integrator:

\[ V_2 = \frac{-V_1}{sRC} \quad \text{and} \quad I_1 = \frac{V_1}{R} \]

\[ V_1 = -sRCV_2 + 0.I_2 \]
\[ I_1 = \frac{V_1}{R} = -sCV_2 + 0.I_2 \]
\[ T = \begin{bmatrix} -sRC & 0 \\ -sC & 0 \end{bmatrix} = sC \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \]

Cascaded integrators:

\[ T_1T_2 = s^2C_0^2 \begin{bmatrix} R^2 & 0 \\ R & 0 \end{bmatrix} = (-sRC_0)^2 \begin{bmatrix} 1 & 0 \\ 1/R & 0 \end{bmatrix} \]

NOTE: B=D=0 means buffered two-port!
Example 5-3 Find $Y$ and $Z$ for the circuit of Fig. 5-7a.

From Fig. 5-7b (where we have chosen for simplicity $V_i = 1 \text{ V}$), by inspection

$$y_{11} \triangleq \frac{I_1^1}{V_1} = I_1^1 = sC_1 + \frac{1}{R} \quad y_{21} \triangleq \frac{I_1^2}{V_1} = I_1^2 = -\frac{1}{R}$$

and from Fig. 5-7c, with $V_2 = 1 \text{ V}$,

$$y_{12} \triangleq \frac{I_2^1}{V_2} = I_2^1 = -\frac{1}{R} \quad y_{22} \triangleq \frac{I_2^2}{V_2} = I_2^2 = sC_2 + \frac{1}{R}$$

Hence

$$Y = \begin{bmatrix} sC_1 + \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & sC_2 + \frac{1}{R} \end{bmatrix}$$

Therefore

$$\Delta_Y = \left( sC_1 + \frac{1}{R} \right) \left( sC_2 + \frac{1}{R} \right) - \frac{1}{R^2} = s^2C_1C_2 + \frac{s(C_1 + C_2)}{R}$$

and, by (5-17),

$$Z = \begin{bmatrix} sC_2 + \frac{1}{R} & \frac{1}{R} \\ \frac{\Delta_Y}{\Delta_Y} & \frac{1}{\Delta_Y} \\ \frac{1}{\Delta_Y} & \frac{sC_1 + \frac{1}{R}}{\Delta_Y} \end{bmatrix}$$

Figure 5-7 Simple $RC$ two-port and the calculation of its admittance parameters.

Of course, $Z$ can also be obtained directly, using the method illustrated in Fig. 5-4. For our example, this gives

$$z_{11} = \frac{V_1^1}{I_1} = \frac{1}{sC_1 + \frac{1}{R + 1/sC_2}}$$

$$z_{11} = \frac{sRC_2 + 1}{s^2RC_1C_2 + s(C_1 + C_2)} = \frac{sC_2 + 1/R}{s^2C_1C_2 + s(C_1 + C_2)/R}$$
Example 5-4 Find \( Z \) and \( Y \) for the two-port shown in Fig. 5-8.

With an input current \( I_1 = 1 \, \text{A} \), from (5-2) and Fig. 5-4b,

\[
z_{11} = V_1^1 = \frac{sL_1 R_1}{sL_1 + R_1} + \frac{R_2/sC}{R_2 + 1/sC} = \frac{s}{s + 1} + \frac{1/s}{1 + 1/s} = 1
\]

and

\[
z_{21} = V_1^2 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1/s}{1 + 1/s} = 1
\]

Similarly, from (5-3) and Fig. 5-4c,

\[
z_{12} = V_2^1 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1}{s + 1}
\]

and

\[
z_{22} = V_2^2 = \frac{sL_2 R_3}{sL_2 + R_3} + \frac{R_2/sC}{R_2 + 1/sC} = 1
\]

Hence,

\[
\Delta_Z = 1 - \frac{1}{(s + 1)^2} = \frac{s^2 + 2s}{s^2 + 2s + 1}
\]

and by (5-19)

\[
y_{11} = \frac{z_{22}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}
\]

\[
y_{12} = y_{21} = -\frac{z_{12}}{\Delta_Z} = -\frac{s + 1}{s^2 + 2s}
\]

\[
y_{22} = \frac{z_{11}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}
\]

Again, \( y_{11} \) will be checked using the scheme of Fig. 5-5b. For \( V_1 = 1 \, \text{V} \),

\[
y_{11} = I_1^1 = \frac{1}{sL_1 R_1 + 1/R_2 + sC + 1/R_3 + 1/sL_2}
\]

\[
= \frac{s}{s + 1} + \frac{1}{2 + s + 1/s} = \frac{s^2 + 2s + 1}{s^2 + 2s}
\]

as before.

![Figure 5-8 RLC two-port example.](image)

Symmetric, reciprocal constant-resistance twoport.
Example 5-5 Find the $z_{ij}$ for the two-port shown in Fig. 5-9.

From Fig. 5-4, we obtain

$$z_{11} = Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}$$

$$z_{12} = z_{21} = \frac{Z_3Z_4}{Z_2 + Z_3 + Z_4}$$

$$z_{22} = \frac{Z_4(Z_2 + Z_3)}{Z_2 + Z_3 + Z_4}$$

The reader should fill in the details.

Hitherto, all two-port examples contained reciprocal circuits. Hence, by (5-5), $z_{12} = z_{21}$ held, and so did

$$y_{12} = y_{21}$$

Equation (5-21) can be obtained either from Eqs. (5-5) and (5-19) or from Fig. 1-6b. For circuits containing active elements, (5-5) and (5-21) need not hold.

Example 5-6 For the circuit of Fig. 5-10a, $Y$ can readily be found using the scheme of Fig. 5-5. With the output port short-circuited and $V_1 = 1$ V (Fig. 5-10b),

$$y_{11} = I_1^1 = \frac{1}{R_1} + \frac{1}{R_2} \quad y_{21} = I_1^1 = -\frac{G}{R_3} - \frac{1}{R_2}$$

whereas if the input port is short-circuited and $V_2 = 1$ V (Fig. 5-10c),

$$y_{12} = I_1^1 = -\frac{1}{R_2} \quad \text{and} \quad y_{22} = \frac{1}{R_3} + \frac{1}{R_2}$$

Here, $y_{12} \neq y_{21}$, except if $G = 0$ or $R_3 \to \infty$, that is, if the active element (which here is a voltage-controlled voltage source) is removed from the circuit.

![Figure 5-10](image-url) (a) Nonreciprocal two-port; (b) and (c) calculation of $Y$ for the two-port.
Example 5-7 For the two-port of Fig. 5-7a, using Fig. 5-11, we find

\[ A = \left( \frac{1}{V_2} \right)_{V_1=1V, I_2=0} = \left( \frac{1/sC_2}{R + 1/sC_2} \right) = sRC_2 + 1 \]

\[ B = \left( -\frac{1}{I_2} \right)_{V_1=1V, I_1=0} = R \]

\[ C = \left( \frac{1}{V_2} \right)_{I_1=1A, I_2=0} = \left( \frac{1}{V_1} \right)_{I_1=1A, I_2=0} \]

\[ = \frac{1}{(1/sC_1)(R + 1/sC_2) + 1/sC_2} = s^2RC_1C_2 + s(C_1 + C_2) \]

\[ = \frac{1}{(1/sC_1 + R + 1/sC_2)R + 1/sC_2} \]

and

\[ D = \left( -\frac{1}{I_2} \right)_{I_1=1A, V_2=0} = \left( R \right)_{sC_1 + 1/R} = sRC_1 + 1 \]

When (5-26) is used in conjunction with the values found earlier for the \( Z \) and \( Y \) of this circuit, the above results can be confirmed.

Check:

Example 5-8 For the circuit of Fig. 5-7a,

\[ \Delta_T = (sRC_2 + 1)(sRC_1 + 1) - R(s^2RC_1C_2 + sC_1 + sC_2) = 1 \]

as expected.
Example 5-9 For the circuit of Fig. 5-10a, the chain parameters can be found from the admittance parameters calculated earlier, for example:

\[
A = \frac{y_{22}}{y_{21}} = -\frac{1}{R_3 + 1/R_2} = \frac{R_2 + R_3}{GR_2 + R_3}
\]

\[
B = -\frac{1}{y_{21}} = -\frac{1}{-G/R_3 - 1/R_2} = \frac{R_2 R_3}{GR_2 + R_3}
\]

\[
C = -\frac{y_{11} y_{22} - y_{12} y_{21}}{y_{21}} = -\frac{y_{11} y_{22}}{y_{21}} + y_{12}
\]

\[
= -\frac{(1/R_1 + 1/R_2)(1/R_3 + 1/R_2)}{-G/R_3 - 1/R_2} - \frac{1}{R_2}
\]

\[
= \frac{(R_1 + R_2)(R_2 + R_3)}{(GR_2 + R_3)R_1 R_2} - \frac{1}{R_2}
\]

\[
D = -\frac{y_{11}}{y_{21}} = -\frac{1/R_1 + 1/R_2}{-G/R_3 - 1/R_2} = \frac{(R_1 + R_2)R_3}{R_1 (GR_2 + R_3)}
\]

Hence

\[
AD - BC = \frac{(R_2 + R_3)(R_1 + R_2)R_3 - R_2 R_3}{R_1} - \frac{(R_1 + R_2)(R_2 + R_3) - GR_2 + R_3}{(GR_2 + R_3)^2}
\]

or, after simplifications,

\[
AD - BC = \frac{R_3}{GR_2 + R_3}
\]

Hence, \(AD - BC = 1\) holds only if \(R_3 \to \infty\) or \(G \to 0\), that is, only if the controlled source is removed from the circuit.

Since Eq. (5-4) can be rearranged six different ways with two parameters on the left-hand side and two on the right-hand side, we can define six different sets of two-port parameters. The reader should consult Ref. 1, table 17-1, for a listing of these parameters and for the formulas needed to convert from one set to another.
Transfer Functions for Terminated Twoports

Transfer Function: Possible output/known input (voltage or current. For a two-port, we may have

Voltage ratio, voltage gain: \( A_v(s) = \frac{V_2(s)}{E(s)} \)

Transfer Admittance: \( Y_T(s) = \frac{I_2(S)}{E(s)} \)

Transfer Impedance: \( Z_T(s) = \frac{V_2(s)}{I(s)} \)

Current Ratio, current gain: \( A_I(s) = \frac{I_2(s)}{I(s)} \)
5-5 TRANSFER FUNCTIONS

When the two-port is excited by a generator and terminated by a load, as illustrated in Fig. 5-1, the signal-transfer properties of the complete circuit can be described by an appropriately chosen transfer function. We define the transfer function as the ratio of an output variable (voltage or current) to a known input quantity (generator voltage or current).† Since most practical generator and load impedances are essentially resistive, we shall restrict our discussions to resistor-terminated reactance two-ports.

In the simplest (and least useful) situation, both terminations are zero or infinite. Then we have one of the four configurations depicted in Fig. 5-18. These circuits are called unterminated (unloaded) two-ports. The proper choice of a transfer function for any of these circuits is obvious and unique. For example, for the circuit of Fig. 5-18a, the output variable must be $V_2$ since $I_2 \equiv 0$; the known input quantity is the generator voltage $E$. Hence, we must choose the voltage ratio $A_V$, defined by

$$A_V(s) \triangleq \frac{V_2(s)}{E(s)}$$  \hspace{1cm} (5-84)

[The reader should keep in mind the dual interpretation of the variable $s$. Thus, for $s = j\omega$, $A_V(j\omega)$ may represent the ratio of the steady-state sine-wave voltage phasors at the output and input. In general, however, $A_V(s)$ is the ratio of the Laplace-transformed output signal $v_2(t)$ and generator signal $e(t)$ for a two-port initially free of stored energy.]

Similarly, for the circuit in Fig. 5-18b the transfer function must be the transfer admittance

$$Y_T(s) \triangleq \frac{I_2(s)}{E(s)}$$  \hspace{1cm} (5-85)

For the circuit of Fig. 5-18c, the transfer function is the transfer impedance

$$Z_T(s) \triangleq \frac{V_2(s)}{I(s)}$$  \hspace{1cm} (5-86)

† Here all quantities are assumed to be functions of $s$, not $t$. 
Finally, for the circuit of Fig. 5-18d, the transfer function is the current ratio

$$A_I(s) \triangleq \frac{I_2(s)}{I(s)}$$  \hfill (5-87)

The transfer functions can readily be calculated from the two-port parameters $Z$ or $Y$ or $T$. For example, for the circuit of Fig. 5-18a, from (5-9)

$$V_1 = z_{11} I_1 + z_{12} I_2 = z_{11} I_1 \quad V_2 = z_{12} I_1 + z_{22} I_2 = z_{12} I_1$$  \hfill (5-88)

Hence

$$A_V = \frac{V_2}{E} = \frac{V_2}{V_1} = \frac{z_{12}}{z_{11}}$$  \hfill (5-89)

Alternatively, from (5-10)

$$I_2 = y_{12} V_1 + y_{22} V_2 = 0$$  \hfill (5-90)

which gives

$$A_V = \frac{V_2}{V_1} = -\frac{y_{12}}{y_{22}}$$  \hfill (5-91)
Or, from (5-24),
\[ V_1 = AV_2 - BI_2 = AV_2 \]  \hspace{1cm} (5-92)
so that
\[ A_V = \frac{V_2}{V_1} = \frac{1}{A} \]  \hspace{1cm} (5-93)

For the circuit of Fig. 5-18b, from (5-9),
\[ V_1 = z_{11}I_1 + z_{12}I_2 = E \quad V_2 = z_{12}I_1 + z_{22}I_2 = 0 \]  \hspace{1cm} (5-94)
Solving (5-94) for \( I_2 \) gives
\[ I_2 = -\frac{Ez_{12}}{z_{11}z_{22} - z_{12}^2} = -\frac{Ez_{12}}{\Delta_Z} \]  \hspace{1cm} (5-95)
so that
\[ Y_T \triangleq \frac{I_2}{E} = -\frac{z_{12}}{\Delta_Z} \]  \hspace{1cm} (5-96)

Alternatively, from (5-10)
\[ I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E \]  \hspace{1cm} (5-97)
so that
\[ Y_T = \frac{I_2}{E} = y_{12} \]  \hspace{1cm} (5-98)

Or, from (5-24),
\[ V_1 = E = AV_2 - BI_2 = -BI_2 \]  \hspace{1cm} (5-99)
which gives
\[ Y_T = \frac{I_2}{E} = -\frac{1}{B} \]  \hspace{1cm} (5-100)

directly.

Exactly analogous manipulations give
\[ Z_T = \frac{V_2}{I} \triangleq z_{12} = -\frac{y_{12}}{\Delta_Y} = \frac{1}{C} \]  \hspace{1cm} (5-101)
for the circuit of Fig. 5-18c and
\[ A_I = -\frac{z_{12}}{z_{22}} = \frac{y_{12}}{y_{11}} = -\frac{1}{D} \]  \hspace{1cm} (5-102)
for the circuit of Fig. 5-18d.

If the two-port has a single resistive termination, it is called a singly terminated or singly loaded two-port. Four possible circuits for such a two-port are illustrated in Fig. 5-19. Notice that two other possible circuits exist which may be obtained by replacing the generator and its internal impedance \( R \) by its Norton equivalent in the circuit of Fig. 5-19b or by its Thevenin equivalent in Fig. 5-19d. Their transfer functions differ only by a factor \( R \) from those of Fig. 5-19b and d, and hence they do not merit separate treatment.
For the circuit of Fig. 5.19a, we can choose \( A_V = V_2/E \) as the transfer function. (Again, a trivially different choice is to select \( Y_T = I_2/E \); since \( I_2 = -V_2/R \), here \( Y_T = -A_V/R \).)

For the circuit of Fig. 5.19b, the transfer function may be selected as \( Y_T = I_2/E \) [or as \( A_I = I_2/I = I_2/(E/R) = RY_T \) if the Norton generator model is substituted].

For the circuit of Fig. 5.19c, we can use \( A_T = I_2/I \) or, as trivial variant, \( Z_T = V_2/I = -I_2 R/I = -RA_I \). For the circuit of Fig. 5.19d, the transfer function can be \( Z_T = V_2/I \) or if the Thevenin equivalent is used for the generator, \( A_V = V_2/E = V_2/(IR) = Z_T/R \).

The transfer functions of the singly loaded two-port can also easily be found in terms of the two-port parameters and \( R \), as will be shown next. For the circuit of Fig. 5.19a, combining the branch relations

\[
V_1 = E \quad V_2 = -RI_2 \quad (5-103)
\]

and the two-port relations (5-9), we get

\[
z_{11} I_1 + z_{12} I_2 = E \quad z_{12} I_1 + (z_{22} + R)I_2 = 0 \quad (5-104)
\]
which gives
\[ I_2 = \frac{z_{12}E}{-z_{11}z_{22} + z_{12}^2 - z_{11}R}, \]  
(5-105)
so that
\[ A_V = \frac{V_2}{E} = \frac{-I_2R}{E} = \frac{z_{12}R}{\Delta z + z_{11}R} \]
(5-106)
Alternatively, from (5-103) and (5-10)
\[ I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E - y_{22}RI_2 \]
(5-107)
which gives
\[ I_2 = \frac{y_{12}E}{1 + y_{22}R} \quad A_V = \frac{-I_2R}{E} = \frac{-y_{12}R}{1 + y_{22}R} \]
(5-108)
Finally, from (5-24), using (5-103), we have
\[ V_1 = E = AV_2 - BI_2 = -ARI_2 - BI_2 \]
\[ I_2 = \frac{-E}{AR + B} \quad A_V = \frac{-I_2R}{E} = \frac{R}{AR + B} \]
(5-109)
Similar calculations performed for the circuit of Fig. 5-19b give
\[ Y_T \triangleq I_2 = \frac{-z_{12}}{\Delta z + z_{22}R} = \frac{y_{12}}{1 + y_{11}R} = \frac{-1}{B + DR} \]
(5-110)
For the circuit of Fig. 5-19c,
\[ A_T \triangleq \frac{I_2}{I} = \frac{-z_{12}}{z_{22} + R} = \frac{y_{12}}{\Delta z R + y_{11}} = \frac{-1}{CR + D} \]
(5-111)
Finally, for the circuit of Fig. 5-19d,
\[ Z_T \triangleq \frac{V_2}{I} = \frac{z_{12}R}{z_{11} + R} = \frac{-y_{22}R}{\Delta y R + y_{22}} = \frac{R}{A + RC} \]
(5-112)
The most important and most widely used circuit is the doubly terminated (or doubly loaded) reactance two-port, illustrated in Fig. 5-20. Depending on whether Thevenin or Norton model is used for the generator and whether \( V_2 \) or \( I_2 \) is used as output variable, any one of the four transfer functions \( A_V, A_I, Z_T, \) and \( Y_T \) can be used to describe the transmission properties of the circuit. If the circuit of Fig. 5-20a is chosen, for example, i.e., a Thevenin generator model, and \( V_2 \) as output variable, \( A_V \) is the proper transfer function. Now the branch relations are
\[ V_1 = E - R_G I_1 \quad V_2 = -I_2 R_L \]
(5-113)
They can be combined with the two-port relations (5-9) to give
\[ (z_{11} + R_G)I_1 + z_{12}I_2 = E \quad z_{12}I_1 + (z_{22} + R_L)I_2 = 0 \]
(5-114)
Solving (5-114) for $I_2$ gives

$$I_2 = \frac{-z_{12}E}{\Delta_z + z_{11}R_L + z_{22}R_G + R_G R_L} \quad (5-115)$$

Hence

$$A_V = \frac{V_2}{E} = \frac{-I_2 R_L}{\Delta_z + z_{11}R_L + z_{22}R_G + R_G R_L} \quad (5-116)$$

Carrying out the calculations in terms of the $y_{ij}$, that is, combining and solving (5-113) and (5-10), gives

$$A_V = \frac{-y_{12}R_L}{\Delta_f R_G R_L + y_{11}R_G + y_{22}R_L + 1} \quad (5-117)$$

Finally, to express $A_V$ in terms of the chain parameters, we combine and solve Eqs. (5-113) and (5-24). This results in

$$A_V = \frac{R_L}{A R_L + B + C R_G R_L + D R_G} \quad (5-118)$$

As will be shown in Chap. 6, the design of a doubly terminated two-port is expediently performed using a different transfer function $H(s)$, which is related to $A_V(s)$ by

$$H(s) \triangleq \frac{1}{2} \sqrt{\frac{R_L}{R_G}} \frac{E}{V_2} = \frac{\sqrt{R_L/R_G}}{2A_V} \quad (5-119)$$
Design of passive two-ports: filter, equalizer (gain or phase).

Delay line

 Specs → Transfer functions → \( Z_{oc}, Y_{sc}, T, \ldots \) → circuit

Example:

Calculate \( Z_T = V_2 / I \) for the circuit shown using its \( z_{ij} \) parameters.

Solution:

Since \( Z_T = z_{12}R / (z_{11} + R) \), we first find

\[
\begin{align*}
  z_{11} &= \frac{(2s^2 + 1)}{s(s^2 + 2)} = 1/2s, s \to 0 \\
  z_{12} &= \left( \frac{V_2}{I_1} \right) \text{ for } I_2 = 0 \to z_{12} = 1/(s^3 + 2s), s \to \infty
\end{align*}
\]

Substituting gives

\[
Z_T = \frac{1}{(s^3 + 2s^2 + 2s + 1)}
\]

Checks: for \( s=0 \), \( Z_T(s) = 1 \); true from circuit diagram. For \( s \to \infty \), \( Z_T(s) \to s^{-3} \), also obvious from circuit.

---

**Figure 5-25** Reactance-ladder realization example.
Example:

Find the $z_{ij}$ and $AI$ for the circuit shown. The results are:

$$z_{12} = \frac{3}{s(s^2 + 2)}$$
$$z_{22} = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$$
$$A_I = \frac{I_2}{I} = -\frac{z_{12}}{z_{22}} = -\frac{3}{(s^2 + 1)(s^2 + 3)}$$

Again, testing for $z=0$ shows that $z_{22} = -z_{12} = 1/(2s/3)$ and $AI=-1$ are correct, as are $z_{22} \rightarrow s$, $z_{12} \rightarrow \frac{3}{s^3}$ and $AI \rightarrow -\frac{3}{s^4}$.