9-1 TOLERANCE ANALYSIS; CIRCUIT SENSITIVITIES

In practice, no matter how accurately and carefully the engineer designs his circuit, the final product will contain imperfect elements which will cause the circuit performance to deviate from the anticipated response. One imperfection will be the inaccuracy of the element values. These values for the actual circuit can lie anywhere in a range, called the tolerance range, which the designer declares acceptable for the purpose. The assignment of tolerances is, in fact, one of the most important parts of the circuit designer's task.

In addition to the unavoidable element-value deviations, or tolerances, other effects, such as aging, temperature and humidity variations, etc., affect the circuit's performance. Also, parasitic elements, e.g., stray capacitances, lead inductances, and winding resistances in inductors, can have appreciable and usually detrimental effects on the circuit response.

In this chapter, some methods will be described which enable the designer to predict the effects of such imperfections. Since the basic question to be answered is how sensitive the circuit is to these perturbations, the process is usually called sensitivity analysis.

Example 9-1 As an illustration, consider the simple circuit shown in Fig. 9-1a. Assume that both resistors have a tolerance range of ±5 percent; that is, the inequalities $47.5 \leq R_1 \leq 52.5$ and $95 \leq R_2 \leq 105$ limit the possible values of the elements. How accurate will $v_0$ be?
Unnormalized sensitivities: change in $v_0$ / change in $R_i$ (Dimension V/Ω).

Alternative: \[ \frac{\Delta v_0}{v_0} / \frac{\Delta R_i}{R_i} \] relative sensitivity.
\[ v_0 = v_{0_{\text{nom}}} + S_1 \Delta R_1 + S_2 \Delta R_2 \]

For “small” \( \Delta R_1, \Delta R_2 \)

\[ \Delta R_2 = 0 \]

\[ \Delta R_1 = 0 \]

Figure 9-2 Linear approximation.

Some measure of the accuracy of (9-4) can be gained by plotting (Fig. 9-2) \( v_0 \) as a function of \( R_1 \) with \( R_2 \) fixed at 100 \( \Omega \) and as a function of \( R_2 \) with \( R_1 \) fixed at 50 \( \Omega \). The relation (9-3) is also plotted as a broken straight line in the figures. The diagrams indicate that the straight-line approximation is valid in the tolerance range. [In fact, the curves give only a rough indication of the accuracy.\(^\dagger\) The correct, but more involved, process is to compare the surface \( v_0 = f(R_1, R_2) \) given by (9-5) on the one hand and the tangent plane given by (9-3) on the other in the tolerance range. The excellent match indicated in Fig. 9-2 makes it more than likely, however, that the accuracy of the linear approximation will be acceptable.]

\(^\dagger\) They show that the constant and linear terms dominate the terms containing \( \partial^i v_0 / \partial R_1^i \), \( \partial^i v_0 / \partial R_2^i \), \( i = 2, 3, \ldots \), in Eq. (9-2). However, they neglect the effect of mixed partial derivatives \( \partial^{i_1 + i_2} v_0 / \partial R_1^{i_1} \partial R_2^{i_2} \).
Having obtained a linear approximation to \( v_0(R_1, R_2) \), it is easy to predict the tolerance effects. The \textit{worst case} occurs when the two terms are of the same sign and \( \Delta R_1 \) and \( \Delta R_2 \) are as large as possible. Hence

\[
|\Delta v_0|_{\text{max}} = \left| \frac{\partial v_0}{\partial R_1} \Delta R_1 \right| + \left| \frac{\partial v_0}{\partial R_2} \Delta R_2 \right| \\
\approx (8.9 \times 10^{-3})(2.5) + (4.44 \times 10^{-3})(5) \approx 0.0444 \text{ V}
\]  
(9-6)

Thus, \( v_0 \) satisfies \( 1.2889 \leq v_0 \leq 1.3777 \) for all permissible values of \( R_1 \) and \( R_2 \).

For \textit{statistical analysis},† the linear approximation also simplifies the calculations. If there is no correlation between the random values assumed by \( R_1 \) and \( R_2 \), and if the variances† of their value distributions are \( \sigma_{R_1}^2 \) and \( \sigma_{R_2}^2 \), then it can be shown (Ref. 1, chap. 7) that the variance \( \sigma_{v_0}^2 \) of \( v_0 \) is

\[
\sigma_{v_0}^2 = \left( \frac{\partial v_0}{\partial R_1} \right)^2 \sigma_{R_1}^2 + \left( \frac{\partial v_0}{\partial R_2} \right)^2 \sigma_{R_2}^2
\]

(9-7)

We can therefore conclude that once an approximate relation of the form (9-3) is obtained, the calculation of tolerance effects becomes an easy task. Hence, the calculation of the partial derivatives \( \frac{\partial v_0}{\partial R_1} \) and \( \frac{\partial v_0}{\partial R_2} \) (often called \textit{sensitivities}) is of crucial importance in tolerance analysis.

It will be shown in Chap. 11 that the \textit{optimization} (automated design) of circuits also requires the circuit sensitivities \( \frac{\partial v_0}{\partial p_i} \), where the \( p_i \)'s are the variable ("designable") parameters of the circuit. These derivatives give an indication of how the \( p_i \) should be changed to improve the performance of the circuit.

For very simple circuits, like those shown in Fig. 9-1a and b, it is possible to calculate the sensitivities analytically. For even moderately large circuits, however, the analytic method becomes very tedious. Since computers are badly suited for analytic differentiation, numerical techniques must be found to accomplish the computation of the sensitivities.

Conceptually, the easiest numerical procedure is to use difference quotients as approximations of the derivatives. For example, one can use the \textit{forward-difference quotient}

\[
\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

\[
\frac{\partial v_0}{\partial R_1} \approx \frac{v_0(R_1 + \Delta R_1, R_2) - v_0(R_1, R_2)}{\Delta R_1}
\]

(9-8)

or the somewhat more accurate \textit{central-difference quotient}

\[
\frac{\partial v_0}{\partial R_1} \approx \frac{v_0(R_1 + \Delta R_1, R_2) - v_0(R_1 - \Delta R_1, R_2)}{2 \Delta R_1}
\]

(9-9)

These approximations are illustrated in Fig. 9-3.

These simple methods have two important disadvantages. The \textit{first} one involves the accuracy of the approximation. If the change in the parameter, say \( \Delta R_1 \), is too small, \( v_0(R_1 + \Delta R_1, R_2) - v_0(R_1, R_2) \) will be the difference of nearly equal quantities and hence inaccurate. If \( \Delta R_1 \) is too large, the difference quotient will be

† The variance \( \sigma_x^2 \) of a quantity \( x \) is defined as the average value of \((\Delta x)^2\) (see Ref. 1, p. 176).
a poor approximation of the derivative. By judicious choice of \(\Delta R_1\) (say, using \(|\Delta R_1/R_1| = 1\) percent) this problem can be overcome. More important is the second disadvantage, which is the large amount of calculation needed to evaluate (9-8) or (9-9). Specifically, if there are \(n\) variable parameters \(p_i\) in the circuit, the calculation of all \(\partial v_0/\partial p_i\) requires \(n + 1\) analyses of the circuit if forward differences are used. This is because the augmented values of \(v_0: v_0(p_1, \Delta p_1, p_2, \ldots, p_n), v_0(p_1, p_2 + \Delta p_2, \ldots, p_n), \ldots, v_0(p_1, p_2, \ldots, p_n + \Delta p_n)\) as well as its nominal value \(v_0(p_1, p_2, \ldots, p_n)\) must all be found. For central differences, the situation is even worse: \(2n\) analyses must be performed. If \(n\) is large (for integrated circuits, it may be of the order of 100 or more), the computational effort and cost may become overwhelming.

In this chapter and the next, a numerical method will be described for the calculation of circuit sensitivities. The method is exact yet well suited for implementation on a digital computer. Most important, it is very economical in terms of computational effort: the calculations required are at most equivalent to two (rather than \(n + 1\)) circuit analyses. Finally, the process can handle all practi-