Reinforcement learning: Markov Decision Processes

CS434
Reinforcement learning
Reinforcement Learning

• You can think of supervised learning as the teacher providing answers (the class labels)
• In reinforcement learning, the agent learns based on a punishment/reward scheme
• Before we can talk about reinforcement learning, we need to introduce Markov Decision Processes
Decision Processes: General Description

• Decide what action to take next given that your action will affect what happens in the future

• Real world examples:
  – Robot path planning
  – Elevator scheduling
  – Travel route planning
  – Aircraft navigation
  – Manufacturing processes
  – Network switching and routing
Sequential Decisions

- Assume a fully observable, deterministic environment, like the example shown here
- Each grid cell is a state
- The goal state is marked +1
- At each time step, agent must move Up, Right, Down, or Left
- How do you get from start to the goal state?
Sequential Decisions

- Suppose the environment is now stochastic
- With 0.8 probability you go in the direction you intend
- With 0.2 probability you move at right angles to the intended direction (0.1 in either direction – if you hit the wall you stay put)
- What is the optimal solution now?
Sequential Decisions

- Up, Up, Right, Right, Right reaches the goal state with probability $0.8^5 = 0.32768$

- But in this stochastic world, going Up, Up, Right, Right, Right might end up with you actually going Right, Right, Up, Up, Right with probability $(0.1^4)(0.8) = 0.00008$

- However, you might end up in the -1 state accidentally
The Effect of Action: Transition Model

- Transition model: specifies the **outcome probabilities for each action in each possible state**
- \( T(s, a, s') = \) probability of going to state \( s' \) if you are in state \( s \) and do action \( a \)
- The transitions are **Markovian**, ie. the probability of reaching state \( s' \) from \( s \) depends only on \( s \) and not on the history of earlier states (aka The Markov Property)
- Mathematically:
  Suppose you visited the following states in chronological order: \( s_0, \ldots, s_t \)
  \[
P(s_{t+1} \mid a, s_0, \ldots, s_t) = P(s_{t+1} \mid a, s_t)
  \]
Markov Property Example

Suppose
\[ s_0 = (1,1), s_1 = (1,2), s_2 = (1,3) \]

If I go *Right* from state \( s_2 \), the probability of going to \( s_3 \) only depends on the fact that I am at state \( s_2 \) and not the entire state history \( \{s_0, s_1, s_2\} \).
The Reward Function

• At each state $s$, the agent receives a reward $R(s)$ which may be positive or negative (but must be bounded)

• Based on reward, we can define the utility of a sequence of state

• For now, we’ll define the utility of a sequence of states as the sum of the rewards received
Reward Function Example

\[ R(4, 3) = +1 \text{ (Agent wants to get here)} \]
\[ R(4, 2) = -1 \text{ (Agent wants to avoid this)} \]
\[ R(s) = -0.04 \text{ (for other states as it takes one step)} \]
\[ U(s_1, ..., s_n) = R(s_1) + ... + R(s_n) \]

If the states an agent goes through are Up, Up, Right, Right, Right, Right, the utility of this state sequence is:

\[-0.04-0.04-0.04-0.04-0.04+1\]
Reward Function Example

If there’s no uncertainty, then the agent would find the sequence of actions that maximizes the sum of the rewards of the visited states.
Markov Decision Process

A sequential decision problem with a fully observable environment with a *Markovian transition model* and *additive rewards* is modeled by a Markov Decision Process (MDP).

An MDP has the following components:

1. A (finite) set of states $S$
2. A (finite) set of actions $A$
3. Transition Model: $T(s, a, s') = P(s' | a, s)$
4. Reward Function: $R(s)$
Example

- State space: (1,1), (1,2) ……(4,3)
- Action space: up, down, right, left
- Transition probability:
  With 0.2 probability you move at right angles to the intended direction (0.1 in either direction – if you hit the wall you stay put)
- Reward:
  R(4,3)=1, R(4,2)=-1, everywhere else: -0.04
Solutions to an MDP

• Why is the following not a satisfactory solution to the MDP?
  
  [1,1]-Up
  [1,2]-Up
  [1,3]-Right
  [2,3]-Right
  [3,3]-Right
Solution to an MDP: Policy

• **Policy**: mapping from a state to an action

• Need to be defined for all states so that the agent will always know what to do

• Notation:
  - $\pi$ denotes a policy
  - $\pi(s)$ denotes the action recommended by the policy $\pi$ for state $s$
Optimal Policy

- Many different policies exist for an MDP
- Some are better than others. The “best” one is called the optimal policy $\pi^*$ (we will define best more precisely in later slides)
- Note: every time we start at the initial state and execute a policy, we get a different state sequence (because the environment is stochastic)
- We get a different utility each time we execute a policy
- Need to measure the expected utility, i.e. the average utility of possible state sequences generated by the policy
Optimal Policy Example

\[ R(s) = -0.04 \]

Notice the tradeoff between risk and reward!
Roadmap for the Next Few Slides

We need to describe how to compute optimal policies

1. Before we can do that, we need to define the utility of a state sequence
2. Before we can do (1), we need to explain the stationarity assumption
3. Before we can do (2), we need to explain finite/infinite horizons
Finite/Infinite Horizons

• Finite horizon: fixed time $N$ after which nothing matters (think of this as a deadline)
• Suppose our agent starts at $(3,1)$, $R(s)=-0.04$, and $N=3$. Then to get to the $+1$ state, agent must go up.
• If $N=100$, agent can take the safe route around

![Grid Diagram](image)

Start here
Nonstationary Policies

- **Nonstationary** policy: the optimal action in a given state changes over time
- With a finite horizon, the optimal policy is nonstationary
- With an infinite horizon, there is no incentive to behave differently in the same state at different times
- With an infinite horizon, the optimal policy is stationary
- We will assume infinite horizons
Stationary Preference

• To calculate the utility of state sequences, you need a stationary preference assumption

• Suppose you have two state sequences:
  \[ s_0, s_1, s_2, \ldots \]
  \[ s_0', s_1', s_2', \ldots \]

• Suppose they begin with the same state \( s_0 = s_0' \)

• Then the two sequences should be preferred in the same order as \( s_1, s_2, \ldots \) and \( s_1', s_2', \ldots \)

• If you prefer one future to another starting tomorrow, you should still prefer that future if it started today
Utility of a State Sequence

Under stationarity, there are two ways to assign utilities to sequences:

1. **Additive rewards:** The utility of a state sequence is:
   \[ U(s_0, s_1, s_2, \ldots) = R(s_0) + R(s_1) + R(s_2) + \ldots \]

2. **Discounted rewards:** The utility of a state sequence is:
   \[ U(s_0, s_1, s_2, \ldots) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots \]
   Where \( 0 \leq \gamma \leq 1 \) is the discount factor
The Discount Factor

- Describes preference for current rewards over future rewards
- Compensates for uncertainty in available time (models mortality)
- Eg. Being promised $10000 next year is only worth 90% of being promised $10000 now
- \( \gamma \) near 0 means future rewards don’t mean anything
- \( \gamma = 1 \) makes discounted rewards equivalent to additive rewards
Utilities

We assume infinite horizons. This means that if the agent doesn’t get to a terminal state, then environmental histories are infinite, and utilities with additive rewards are infinite. How do we deal with this? Discounted rewards makes utility finite. Assuming largest possible reward is $R_{max}$ and $\gamma < 1$,

$$U(s_0, s_1, s_2, \ldots) = \sum_{t=0}^{\infty} \gamma^t R(s_t)$$

$$\leq \sum_{t=0}^{\infty} \gamma^t R_{max} = \frac{R_{max}}{(1 - \gamma)}$$
Computing Optimal Policies

• A policy \( \pi \) generates a sequence of states
• But the world is stochastic, so a policy \( \pi \) has a range of possible state sequences, each of which has some probability of occurring
• The value of a policy is the expected sum of discounted rewards obtained by following this policy
The Optimal Policy

• Given a policy $\pi$, we write the expected sum of discounted rewards obtained as:

$$E\left[ \sum_{t=0}^{\infty} \gamma^t R(s_t) \mid \pi \right]$$

• An optimal policy $\pi^*$ is the policy that maximizes the expected sum above

$$\pi^* = \arg \max_\pi E\left[ \sum_{t=0}^{\infty} \gamma^t R(s_t) \mid \pi \right]$$
The Optimal Policy

- For every MDP, there exists an optimal policy
- There is no better option (in terms of expected sum of discounted rewards) than to follow this policy
- How do you calculate this optimal policy? Can’t evaluate all policies...too many of them
- Instead, we calculate the *utility of the states*
- Then use the state utilities to select an optimal action in each state
Rewards vs Utilities

• What’s the difference between $R(s)$ the reward for a state and $U(s)$ the utility of a state?
  – $R(s)$ – the short term reward for being in $s$
  – $U(s)$ – The long-term total expected reward for the sequence of states starting at $s$ (not just the reward for state $s$)
Utilities in the Maze Example

Start at state (1,1). Let’s suppose we choose the action Up.

\[ U(1,1) = R(1,1) + \ldots \]

Reward for current state
Utilities in the Maze Example

Start at state (1,1). Let’s choose the action Up.

\[ U(1,1) = R(1,1) + 0.8 \cdot U(1,2) + 0.1 \cdot U(2,1) + 0.1 \cdot U(1,1) \]
Utilities in the Maze Example

Now let’s throw in the discounting factor

\[ U(1,1) = R(1,1) + \gamma \times 0.8 \times U(1,2) + \gamma \times 0.1 \times U(2,1) + \gamma \times 0.1 \times U(1,1) \]
The Utility of a State

If we choose action $a$ at state $s$, expected future rewards (discounted) are:

$$U_a(s) = R(s) + \gamma \sum_{s'} T(s, a, s') U(s')$$

- Reward at current state $s$
- Probability of moving from state $s$ to state $s'$ by doing action $a$
- Expected sum of total discounted rewards starting at state $s'$ by taking action $a$
- Expected sum of future discounted rewards starting at state $s'$
The Utility of a State

- In the previous example, we define the utility assuming that action $a$ is taken at state $s$.
- What we want is the utility of a state $s$ assuming that we can choose the optimal action to take at state $s$.
- We modify the previous formula slightly by adding a max term over actions.

$$U_a(s) = R(s) + \gamma \sum_{s'} T(s, a, s') U(s')$$

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$$
The Utility of a State

The utility of a state is the immediate reward for that state plus the expected discounted utility of the next state, assuming the agent chooses the optimal action.

\[ U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s') \]

This is called the Bellman Equation.
The Optimal Policy

• Selection of the action \( \pi^*(s) = a \) which maximizes the expected utility \( U(s') \)

\[
\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') U(s')
\]

• Intuitively, \( \pi^* \) gives us the best action we can take from any state to maximize our future discounted rewards
What You Should Know

• How to formulate a problem as an MDP
• What the Markov property is
• How to calculate the utility for a state in an MDP
• What the Bellman equation is