1. Convert the following NFAs from HW3 to DFA:

(a) \(\{ab, aba\}^*\)
(b) bitstrings with 0 as the third last symbol
(c) bitstrings that contain 0100

How do these converted DFAs compare to your own DFAs in HW3?

**Solution:**

(a) NFA:

conversion:

\[
\begin{array}{c|cc}
 & a & b \\
 A = \{q_0\} & B & B \\
 B = \{q_1, q_2\} & C & C \\
 C = \{q_0, q_3\} & D & C \\
 D = \{q_0, q_1, q_2\} & B & C \\
\end{array}
\]

DFA:
(b) NFA:

\begin{align*}
A &= \{q_0\} \\
B &= \{q_0, q_1\} \\
C &= \{q_0, q_1, q_2\} \\
D &= \{q_0, q_2\} \\
E &= \{q_0, q_1, q_2, q_3\} \\
F &= \{q_0, q_2, q_3\} \\
G &= \{q_0, q_1, q_3\} \\
H &= \{q_0, q_3\}
\end{align*}

DFA:

\begin{align*}
\begin{array}{c|cc}
0 & 1 \\
\hline
A & B & A \\
B & C & D \\
C & E & F \\
D & G & H \\
E & G & H \\
F & C & D \\
G & B & A \\
\end{array}
\end{align*}
(c) NFA:

\[
\begin{align*}
A &= \{q_0\} \\
B &= \{q_0, q_1\} \\
C &= \{q_0, q_2\} \\
D &= \{q_0, q_1, q_3\} \\
E &= \{q_0, q_1, q_4\} \\
F &= \{q_0, q_2, q_4\} \\
G &= \{q_0, q_1, q_3, q_4\} \\
H &= \{q_0, q_4\}
\end{align*}
\]

DFA:

\[
\begin{array}{c|cc}
& 0 & 1 \\
A & B & A \\
B & C & B \\
C & D & A \\
D & E & C \\
E & F & E \\
F & G & H \\
G & E & F \\
H & E & H
\end{array}
\]

Hint: this DFA is not likely to be same as your own DFA in HW3; 4 acceptance states could be merged to one.
2. For any given epsilon-free NFA $M = (Q, \Sigma, \delta, q_0, F)$,

(a) construct a DFA $M'$ such that $L(M) = L(M')$.

(b) prove that $L(M) = L(M')$.

i.e., for all $w \in L(M)$, you want to show $w \in L(M')$, and for all $w \in L(M')$, you want to show $w \in L(M)$.

Hint: first prove a lemma by induction:
$
\forall q \in Q, \forall w \in \Sigma^*, \delta^*(q, w) = \delta^*(\{q\}, w).
$

Solution:

(also see slides “week3 (NFA)”, page 12)

(a) $M' = (Q', \Sigma, \delta', q'_0, F')$

where

$Q' = P(Q)$,
$q'_0 = \{q_0\}$,
$F' = \{A \in P(Q) | A \cap F \neq \emptyset\}$,
$\delta'(R, a) = \cup_{r \in R} \delta(r, a)$.

(b) Proof:

Lemma: $\delta^*(q, w) = \delta^*(\{q\}, w)$.

Proof of Lemma: Do induction on $|w|$.

Base case: when $w = \epsilon$, $\delta^*(q, \epsilon) = \{q\} = \delta^*(\{q\}, \epsilon)$.

Inductive case: assume lemma holds for $|w| \leq n, n \geq 0$. When $|w| = n + 1$, denote $w = xb, b \in \Sigma$. Then

\[
\delta^*(\{q\}, xb) \\
= \delta'(\delta^*(\{q\}, x), b) \\
= \delta'(\delta^*(q, x), b) \\
= \cup_{r \in \delta'(q, x)} \delta(r, b) \\
= \delta^*(q, xb)
\]

(by I.H.)

i.e., lemma holds for $|w| = n + 1$.

By lemma, we have $L(M) = L(M')$ because

$\forall w \in \Sigma^*$,

$w \in L(M) \\
\iff \delta^*(q_0, w) \cap F \neq \emptyset \\
\iff \delta^*(\{q_0\}, w) \cap F \neq \emptyset \\
\iff \delta^*(\{q_0\}, w) \in F' \\
\iff w \in L(M')$

Cont.
3. Show that for an NFA w/o epsilons, the two definitions of $\delta^*$ are equivalent, i.e., if we use the standard definition ($w = xa$), you want to show:

$$\delta^* (q, ax) = \bigcup_{p \in \delta(q,a)} \delta^* (p, x)$$

**solution:**

Prove by induction on $|w|$.

Base Case: when $w = a, a \in \Sigma$,

$$\delta^* (q, a)$$

$$= \bigcup_{p \in \delta^* (q, \epsilon)} \delta (p, a)$$

$$= \delta (q, a)$$

$$= \bigcup_{p \in \delta(q, a)} p$$

$$= \bigcup_{p \in \delta(q, a)} \delta^* (p, \epsilon)$$

Inductive Case: assume theorem holds for $|w| \leq n, n \geq 1$.

When $|w| = n + 1$, denote $|w| = axb, a, b \in \Sigma$.

$$\delta^* (q, axb)$$

$$= \bigcup_{p \in \delta^* (q, ax)} \delta (p, b)$$

$$= \bigcup_{p \in \bigcup_{p' \in \delta(q, a)} \delta^* (p', x)} \delta (p, b)$$

(by I.H.)

$$= \bigcup_{p' \in \delta(q, a)} \bigcup_{p \in \delta^* (p', x)} \delta (p, b)$$

$$= \bigcup_{p' \in \delta(q, a)} \delta^* (p, xb)$$

i.e., theorem holds for $|w| = n + 1$. 

Cont.
4. Write at least three definitions of epsilon-closure. Also define the epsilon-closure of a set of states.

**solution:**

(also see slides “week4”, page 2)
def 0: $E(q) = \{ p | p \text{ is reachable from } q \text{ by 0 or more } \epsilon \text{ edges } \}$.
def 1: $E(q)$ is the smallest set s.t.
- $q \in E(q)$
- if $q \in E(q)$, then $\delta(p, \epsilon) \subseteq E(q)$.
def 2: $E(q) = \bigcup_i E_i(q)$
- $E_0(q) = \{ q \}$
- $E_{i+1}(q) = \bigcup_{p \in E_i(q)} \delta(p, \epsilon)$
closure of a set of states: $E(R) = \bigcup_{q \in R} E(q)$.
5. (Redo 2 for epsilons) For any given NFA with epsilons \( M = (Q, \Sigma, \delta, q_0, F) \),

   (a) construct a DFA \( M' \) such that \( L(M) = L(M') \).

   (b) prove that \( L(M) = L(M') \). i.e., \( \forall w \in L(M) \), you want to show \( w \in L(M') \), and \( \forall w \in L(M') \),

      you want to show \( w \in L(M) \).

solution:

(a) 

\[
M' = (Q', \Sigma, \delta', q'_0, F'),
\]

where

\[
Q' = P(Q),
\]

\[
q'_0 = \{q_0\},
\]

\[
F' = \{A \in P(Q) | A \cap F \neq \emptyset\},
\]

\[
\delta'(R, a) = E(\cup_{r \in R} \delta(r, a)), a \in \Sigma.
\]

(b) Proof:

Lemma: \( \delta^*(q, w) = \delta^*({q}, w) \).

Proof of Lemma: Do induction on \( |w| \).

Base case: when \( w = \epsilon \), \( \delta^*(q, \epsilon) = E({q}) \), \( \delta^*({q}, \epsilon) = E(\cup_{r \in {q}} \delta(r, \epsilon)) = E(\delta(q, \epsilon)) = E({q}) \),

   i.e. lemma holds.

Inductive case: assume lemma holds for \( |w| \leq n, n \geq 0 \). When \( |w| = n + 1 \), denote \( w = xb, b \in \Sigma \).

Then

\[
\delta^*({q}, xb) = \delta'(\delta^*({q}, x), b)
\]

\[
= \delta'(\delta^*({q}, x), b)
\]

\[
= E(\cup_{r \in \delta'(q, x)} \delta(r, b))
\]

\[
= \delta^*({q}, xb)
\]

   i.e., lemma holds for \( |w| = n + 1 \).

By lemma, we have \( L(M) = L(M') \) because

\[
\forall w \in \Sigma^*,
\]

\[
w \in L(M) \iff \delta^*({q_0}, w) \cap F \neq \emptyset
\]

\[
\iff \delta^*({q_0}, w) \cap F \neq \emptyset
\]

\[
\iff \delta^*({q_0}, w) \in F'
\]

\[
\iff w \in L(M')
\]

(Hint 1: this definition of \( \delta' \) is equivalent as the one on textbook (since \( E(\cup_i S_i) = \cup_i (E(S_i)) \)).

(Hint 2: From the definition of \( E \), we can prove \( E(\cup_i S_i) = \cup_i (E(S_i)) \) and \( E(E(S)) = E(S) \). )

Cont.
6. (Redo 3 for epsilons) Figure out an alternative definition of $\delta^*$ for NFA with epsilons, and prove the equivalence.

**solution:**

Alternative definition of $\delta^*$ on $|w| > 0$:

$$
\delta^*(q, w) = \bigcup_{p \in E(\delta(q, a))} \delta^*(p, x), w = ax, a \in \Sigma.
$$

Now prove for the original definition of $\delta^*$, and $|w| > 0$, we have

$$
\delta^*(q, w) = \bigcup_{p \in E(\delta(q, a))} \delta^*(p, x), w = ax, a \in \Sigma.
$$

Prove by induction on $|w|$.

Base Case: when $w = a, a \in \Sigma$,

\[
\delta^*(q, a) = E(\bigcup_{p \in E(\delta(q, a))} \delta(p, a)) = E(\delta(q, a))
\]

\[
\cup_{p \in E(\delta(q, a)))} \delta^*(p, \epsilon) = \cup_{p \in E(\delta(q, a))} E(\{p\}) = E(\cup_{p \in E(\delta(q, a))} \{p\}) = E(E(\delta(q, a))) = E(\delta(q, a))
\]

i.e., theorem holds $w = a$.

Inductive Case: assume theorem holds for $|w| \leq n, n \geq 1$.

When $|w| = n + 1$, denote $|w| = axb, a, b \in \Sigma$.

\[
\delta^*(q, axb) = E(\bigcup_{p \in E(\delta(q, ax))} \delta(p, b)) = E(\bigcup_{p \in E(\delta(q, ax))} (\bigcup_{p' \in E(\delta(q, a))} \delta(p', x)) \delta(p, b)) = E(\bigcup_{p' \in E(\delta(q, a))} (\bigcup_{p \in E(\delta(q, a))} \delta(p', x)) \delta(p, b)) = E(\bigcup_{p' \in E(\delta(q, a))} \delta^*(p, xb))
\]

i.e., theorem holds for $|w| = n + 1$. 

Cont.
7. Devise an algorithm to convert an NFA with epsilons to an epsilon-free NFA.

solution:

Algorithm 1 Algorithm for converting an NFA to an ε-free NFA

Input: an NFA \( M = \{ Q, \Sigma, \delta, q_0, F \} \)

Output: an NFA \( M' = \{ Q', \Sigma, \delta', q'_0, F' \} \), \( \delta' : Q \times \Sigma \rightarrow P(Q) \)

1: repeat
2: \( \text{flag} := \text{False} \)
3: for \( q \in Q \) do
4: if \( \text{flag} = \text{True} \) then
5: break
6: end if
7: \( S := E(q) \)
8: if \( S \neq \{ q \} \) then
9: for \( a \in \Sigma \) do
10: \( \delta(q, a) := \cup_{p \in S} \delta(p, a) \)
11: end for
12: \( \delta(q, \epsilon) := \delta(q, \epsilon) - S \)
13: if \( S \cap F \neq \emptyset \) then
14: \( F := F \cup \{ q \} \)
15: end if
16: \( \text{flag} := \text{True} \)
17: end if
18: end for
19: until \( \text{flag} = \text{False} \)
20: return \((Q, \Sigma, \delta, q_0, F)\)

(Hint: a faster version would be processing all the states in \( S \) in one loop, instead of processing \( q \) only. (from line 8 to line 17))
8. Convert the NFAs from the last two questions in Quiz 3 to DFAs. Quiz 3 solutions are on canvas.

**solution:**

(a) NFA:

![NFA Diagram]

conversion:

<table>
<thead>
<tr>
<th>$A = {q_0, q_1, q_4}$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = {q_2, q_5}$</td>
<td>$C$</td>
</tr>
<tr>
<td>$C = {q_3, q_4}$</td>
<td>$D$</td>
</tr>
<tr>
<td>$D = {q_1, q_5}$</td>
<td>$E$</td>
</tr>
<tr>
<td>$E = {q_2, q_4}$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F = {q_3, q_5}$</td>
<td>$G$</td>
</tr>
<tr>
<td>$G = {q_1, q_4}$</td>
<td>$C$</td>
</tr>
</tbody>
</table>

DFA:

![DFA Diagram]

(Hint: states $A$ and $G$ could be merged here.)
(b) NFA:

\[
\begin{align*}
A &= \{q_0\} & \text{a} \\
B &= \{q_1, q_3\} & \text{B} \\
C &= \{q_2, q_3, q_4\} & \text{C} \\
D &= \{q_0, q_3, q_4\} & \text{D} \\
E &= \{q_1, q_3, q_4\} & \text{E}
\end{align*}
\]

DFA:

(Hint: states \(B, C, D\) and \(E\) could be merged here.)
9. Why it is important to add “the smallest set” in Def. 1 of epsilon-closure? Give an example where dropping “the smallest” doesn’t make sense.

**solution:**

“the smallest set” guarantees that all states in $E(q)$ would be reachable from $q$ by only using zero to many numbers of $\epsilon$.

Consider NFA in question 8 (b). If “the smallest” is dropped, $q_1, q_3, q_4$ satisfies the definition of $E(q_1)$. However, $q_1$ cannot reach $q_4$ by only using any number of $\epsilon$. 
10. For each of the following NFAs, do
   (a) compute epsilon-closure for each state,
   (b) convert it to a DFA,
   (c) explain the intuitive meaning of the language
   (d) try to find a smaller but equivalent DFA, if any.

   ![Diagram of NFA]

   Now flip the two links between $s$ and $v$ and redo everything. (i.e., from $s$ on $a$ goes to $v$, from $v$ on $b$ goes back to $s$).
   Now further add an $\epsilon$ link from $t$ to $r$ and redo everything.
solution:

(a1)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$E(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>${q, r, s, t}$</td>
</tr>
<tr>
<td>$r$</td>
<td>${r, t}$</td>
</tr>
<tr>
<td>$s$</td>
<td>${s, t}$</td>
</tr>
<tr>
<td>$t$</td>
<td>${t}$</td>
</tr>
<tr>
<td>$v$</td>
<td>${v}$</td>
</tr>
</tbody>
</table>

(b1)

conversion:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = {q, r, s, t}$</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>$B = {r, t, v}$</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>$C = {s, t}$</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>$D = {r, t}$</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>$E = {v}$</td>
<td></td>
<td>C</td>
</tr>
</tbody>
</table>

DFA:

(c1)

This languages contains strings with the format $b^*$ or strings with the format $(ba)^*$, but not other strings.

(d1)

DFA in (b1) is the smallest.
(a2)
same as (a1).

(b2)
conversion:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>q, r, s, t</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>v</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>r, t</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>s, t</td>
<td>B</td>
</tr>
</tbody>
</table>

DFA:

(c2)
This language contains strings with the format $b^*$ or strings with the format $(ab)^*$, but not other strings.

(d2)
DFA in (b2) is the smallest.
<table>
<thead>
<tr>
<th>x</th>
<th>E(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>{q, r, s, t}</td>
</tr>
<tr>
<td>r</td>
<td>{r, t}</td>
</tr>
<tr>
<td>s</td>
<td>{r, s, t}</td>
</tr>
<tr>
<td>t</td>
<td>{r, t}</td>
</tr>
<tr>
<td>v</td>
<td>{v}</td>
</tr>
</tbody>
</table>

(a3)

(b3)

conversion:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = {q, r, s, t}</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>B = {v}</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>C = {r, t}</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>D = {r, s, t}</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

DFA:

(c3)

This language contains strings with the format \((ab)^*b^*\), but not other strings.

(d3)

State A and D could be merged.

DFA: