Quiz 2 much better than Quiz 1

- quiz 2 mean: 2.47; quiz 1 mean: 2.0
Theorem

The class of regular languages is closed under concatenation.

If $L_1$ and $L_2$ are regular, then so is $L_1 \circ L_2$.

Proof

Can't do it yet.
We need...

Nondeterminism

- concatenation of regular languages is still regular
Non-Determinism

"Given the current state, there may be multiple next states."

- The next state is chosen at random.
- All next states are chosen in parallel and pursued simultaneously.

FSM means Deterministic
Finite State Automaton/Machine

DFA means DETERMINISTIC
Finite State Automaton

NFA means NONDETERMINISTIC
Finite State Automaton

Now we will allow
- Multiple edges with the same label out of a node.

Epsilon Edges

Which edge should you take???

Can take an ε-edge without scanning a symbol. It is "OPTIONAL"!
Any path to accept is accepted

- DFA is much harder (see HW1)
- but does NFA help for the complement?

**EXAMPLE**

All strings that contain 011110

**EXAMPLE STRING:** 0100 011110101

Lots of bad choices
that don't work,
that don't reach an accept state

All we need is one way
to reach ACCEPT.

*If there is any way to run the machine that ends with ACCEPT, then the NFA accepts.*
As long as there is (at least) one path...

Lots of choices - which one to try?

* Try them all.
* Make the right choice at each point.

Deterministic

Non-deterministic

Choice points

Accept (or not)

Choice points

Reject

Accept

Just need one accept!

Example

String:
010110

"Computation Tree" "Choice Tree"
NFA == DFA, but often a lot easier

**Theorem**
For every nondeterministic F.S.M., there is an equivalent deterministic F.S.M.

... But it may be large and hard to find!

**Example**
All strings over \( \{0,1\}^* \) that have a “0” in the second to the last position.

**NFA**

Q: how to use NFA for union? for intersection?

**Example**
String contains either ...
... 0100 ...
or ... 0111 ...

When to start looking? Which string to look for?

Non-determinism!

**CHALLENGE:**
Build/Design a DFA to recognize this language.
Power set (all subsets of a set)

- Power set is one of the most important & beautiful mathematical concepts.
- \( P(Q) \) is often written as \( 2^Q \) since \( |2^Q| = 2^{|Q|} \).

\[
2^Q \triangleq \{ A \mid A \subseteq Q \}
\]
Q: is every DFA also an NFA??
A: technically, no!
\[
\delta_{NFA}(p, a) = \{\delta_{DFA}(p, a)\}
\]
Computation of NFA w/o epsilon

- string \( w \) is accepted iff. \( \delta^*(q_0, w) \cap F \neq \emptyset \)
- without epsilon transitions, \( \delta^* \) is easy to define
- with epsilon transitions, you need to define epsilon-closure (very hard; will come back later)

\[
\delta^*(q, \varepsilon) = \{q\}
\]

\[
\delta^*(q, xa) = \bigcup_{p \in \delta^*(q,x)} \delta(p, a)
\]

alternatively

\[
\delta^*(q, ax) = \bigcup_{p \in \delta(q,a)} \delta^*(p, x)
\]

\[
\bigcup_{t \in \{1,2,3\}} \{t, t^2\} = \{1,1\} \cup \{2,4\} \cup \{3,9\}
\]
NFA => DFA w/o epsilon

how to convert NFA to DFA?

EXAMPLE
Accept all strings over \{0,1,\}^* ending with "00."

String to check: 00100
Simulate the execution. Put a finger on any state we could be in.

\emptyset A B C AB BC AC ABC

Let \( N \) = Number of states in NFA. What is the (worst case) number of states in the equivalent DFA?

unused states
NFA => DFA w/o epsilon

THEOREM

Every Nondeterministic FSM has an equivalent Deterministic FSM.

"Equivalent" = Recognizes the same language.

PROOF BY CONSTRUCTION

Given a NFA, let's show how to build an equivalent DFA.

Let $M = (Q, \Sigma, \delta, q_0, F)$

Construct $M' = (Q', \Sigma, \delta', q'_0, F')$

where...

$Q' = P(Q)$

Assume NFA has $k$ states.

Then the DFA will have $2^k$ states.

$\text{A B C} \rightarrow \text{A B C AB BC AC ABC}$
Formally: subset construction

NFA $M = (Q, \Sigma, \delta, q_0, F)$

DFA $M' = (Q', \Sigma, \delta', q_0', F')$

where $Q' = P(Q)$

$q_0' = \{q_0\}$

$F' = \{A \in P(Q) \mid A \cap F \neq \emptyset\}$

$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$

Thm.: $L(M) = L(M')$

Idea: $\forall w \in L(M), \exists p \in F$, s.t. $\delta^*(q_0, w) = p$.

want to show: $p \in \delta'(q_0, w)$

also need the other direction
**Epsilon-Closure**

**But what about ε-edges?**

Consider a state in the DFA that we're building.

A state "R" in the DFA is a set of states from the NFA.

Look back at M, the NFA. What states can we reach by going through ε-edges?

Also include the states we're in (i.e., include B, C, and E).

**Define "Epsilon-closure"**

\[ E(R) = \{ q \in Q \mid q \text{ can be reached from a state in } R \text{ by following zero or more } \varepsilon \text{-edges.} \} \]

**Example**

\[ E([BCE]) = \{ B, C, D, E, G, H \} = \{BCDEGH\} \]

**Modify the transition function:**

\[ S'(R, a) = \{ q \in Q \mid q \in E(S(r, a)) \} \]

for some \( r \in R \)

Also, modify the start state in the constructed DFA:

\[ q_0' = E([\varepsilon, \varepsilon]) \]

**End of proof**