Step response:

\[ v[n] \xrightarrow{H} s[n] \xleftarrow{\text{step response}} \]

\[ v(t) \xrightarrow{H} s(t) \]

Example:

\[ h[n] = e^n v[n] \]

Determine the step response \( s[n] \).

\[ s[n] = v[n] * h[n] = \sum_{k=-\infty}^{\infty} v[n-k] h[k] \]

\[ = \sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{n} e^k v[k] \]

\[ = \left\{ \begin{array}{ll}
1 - \frac{1}{e} & \text{if } n \geq 0 \\
0 & \text{if } n < 0
\end{array} \right. \]

**Differential & Difference Equation Representations of LTI Systems**

\[ x(t) \rightarrow [H] \rightarrow y(t) \]

\[ x[n] \quad y[n] \]

Input-output relation can be described as

\[ \text{CT} \quad \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \quad \text{Linear constant-coefficient differential equation.} \]

\[ \text{DT} \quad \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{Linear constant-coefficient difference equation.} \]

- \( a_k, b_k \) constants
- Order of differential/difference equation: \((N,M)\)
- Often \( h \geq M \), and order is described using \( N \) only
2nd-order difference equation

\[ y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n] + 2x[n-1] \]

\[ \begin{cases} (a_0, a_1, a_2) = (1, 1, 1/4) \\ (b_0, b_1) = (1, 2) \end{cases} \]

\[ y[n] \] can be evaluated recursively:

Assume \( y[-1] = 1 \), \( y[-2] = -2 \)

\[ y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4} y[-2] \]
\[ = 1 + 0 -1 + \frac{1}{2} = \frac{1}{2} \]

\[ y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4} y[-1] \]
\[ = \frac{1}{2} + 2 - \frac{1}{2} - \frac{1}{4} = \frac{3}{4} \]

\[ \vdots \]

we don't want to do this manually, take too long.

---

Describe the following RLC circuit by a differential equation.

\[ x(t) \quad \phi \quad R \rightarrow \quad L \rightarrow \quad y(t) \quad \phi \quad C \]

\[ Ry(t) + L \frac{d}{dt} y(t) + \frac{1}{c} \int_0^t y(\tau) d\tau = x(t) \]

\[ \frac{1}{c} y(t) + R \frac{d}{dt} y(t) + L \frac{d^2}{dt^2} y(t) = \frac{d}{dt} x(t) \]

\[ a_0 = \frac{1}{c}, \quad a_2 = R, \quad a_2 = L \]

\[ b_0 = 0, \quad b_1 = 1 \]

\[ N = \mathbb{R} \]
SOLVING DIFFERENTIAL AND DIFFERENCE EQUATIONS

Just a review, rather than in depth:

Complete Solution

Solution forms: Homogeneous Particular

CT

$$y(t) = y^{(h)}(t) + y^{(p)}(t)$$

DT

$$y[n] = y^{(h)}[n] + y^{(p)}[n]$$

Natural response

Input signal $x(t)$ or $x[n] = 0$

Depends on initial condition!

Forced response

Initial rest

Depends on input signal!

GENERAL CASE

Homogeneous Solution

CT system: $y^{(h)}(t)$ is the solution of the homogeneous equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0$$

The homogeneous solution is of the form:

$$y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t}$$

where $r_i$ are the $N$ roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r_i^k = 0$$

Note: $c_i$: to be determined (later) so that the complete solutions satisfy the initial conditions.
DT system: \( y^{(k)}[n] \) is the solution of the homogeneous equation:

\[
\sum_{k=0}^{N} \alpha_k y^{(k)}[n-k] = 0
\]

The homogeneous solution is of the form: \( y^{(k)}[n] = \sum_{i=1}^{N} c_i r_i^n \)

where \( r_i \) are the \( N \) roots of the system's characteristic equation

\[
\sum_{k=0}^{N} a_k r^{N-k} = 0
\]

Note:
- \( c_i \): same as CT case.
- CT and DT characteristic equations are different.

Example

- Homogeneous equation: \( \frac{1}{2} y[n] - \frac{1}{4} y[n-1] = 0 \) (set \( x[n]=0 \))

\[ a_0 = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases} \]

\[ a_1 = -\frac{1}{4} \]

\[ N = 1 \]

\[
(1) \cdot y^1 + (-\frac{1}{4}) \cdot y^0 = 0 \Rightarrow y = \frac{1}{4}
\]

- Characteristic equation:

\[
\sum_{k=0}^{N} a_k r^{N-k} = 0
\]

- Solution of homogeneous equation: \( y^{(k)}[n] = c_1 (\frac{1}{4})^n \)

\( c_1 \): to be determined so that the complete solutions satisfy the initial conditions.
\[ \frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2 y(t) = x(t) + \frac{dx(t)}{dt} \]

Find \( y'(t) \),

\[ N = 2, \quad a_0 = 2, \quad a_1 = 3, \quad a_2 = 1 \]
\[ \sum_{k=0}^{N} a_k r^k = 0 \]
\[ 2 + 3r + r^2 = 0 \]
\[ (r+1)(r+2) = 0 \]
\[ r_1 = -1, \quad r_2 = -2 \]

\[ y(t) = c_1 e^{-t} + c_2 e^{-2t} \]

- A particular solution is assumed independent of the homogeneous solution.
- Usually obtained assuming that output has the same form as input signals.
- The form of the particular solution associated with common inputs are summarized in the following table.

<table>
<thead>
<tr>
<th>Continuous time</th>
<th>Discrete time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>Particular solution</td>
</tr>
<tr>
<td>l</td>
<td>c</td>
</tr>
<tr>
<td>t</td>
<td>( c_1 t + c_2 )</td>
</tr>
<tr>
<td>( e^{\alpha t} )</td>
<td>( c e^{\alpha t} )</td>
</tr>
<tr>
<td>( \cos(\omega t + \phi) )</td>
<td>( c_1 \cos(\omega t + \phi) + c_2 \sin(\omega t + \phi) )</td>
</tr>
</tbody>
</table>

INPUT~ PARTICULAR FORM
Example

Assume a particular solution of the form

\[ y^{(p)}[n] = c_p \left( \frac{1}{2} \right)^n \]  (for input in the form of \( \alpha^n u[n] \))

\[ c_p \left( \frac{1}{2} \right)^n - \frac{1}{4} c_p \left( \frac{1}{2} \right)^{n-1} = \left( \frac{1}{2} \right)^n \]

\[ c_p - \frac{1}{4} c_p \left( \frac{1}{2} \right) = 1 \]

\[ c_p = \frac{2}{3} \Rightarrow c_p = 2 \]

\[ y^{(p)} = 2 \left( \frac{1}{2} \right)^n \]