6. Impulse function

\[
\delta(t) = \begin{cases} 
0, & t \neq 0 \\
\int_{-\infty}^{\infty} \delta(t) \, dt = \int_{0^-}^{0^+} \delta(t) \, dt = 1 
\end{cases}
\]

\[
\delta[n] = \begin{cases} 
1, & n = 0 \\
0, & n \neq 0 
\end{cases}
\]

- Let \( x_{\Delta}(t) \) be a rectangle with area one and width \( \Delta \) and height \( a \). Then, \( \delta(t) = \lim_{\Delta \to 0} ax_{\Delta}(t) \).

\[
\delta(-t) = \delta(t) \quad \Delta \leq -\tau \leq 0 \Rightarrow \sigma(-\tau) = \sigma(0) = \sigma(0^+) = 1
\]

\[
\delta(-t) = \delta(t) \quad \Delta \leq t \leq 0 \Rightarrow \Delta \leq \tau < 0
\]

\[
\delta(t) = \frac{du(t)}{dt} \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau) \, d\tau
\]

\[
\delta(at) = \frac{1}{a} \delta(t), \quad a > 0
\]

\[
\int_{-\infty}^{\infty} x(t) \delta(t - t_0) \, dt = x(t_0)
\]

\[
\int_{-\infty}^{\infty} x(t) \frac{d}{dt} \delta(t - t_0) \, dt = \frac{d}{dt} x(t) \big|_{t = t_0} \quad (p50 \text{ of text})
\]

\[
\int_{-\infty}^{\infty} x(t) \frac{d^n}{dt^n} \delta(t - t_0) \, dt = \frac{d^n}{dt^n} x(t) \big|_{t = t_0}
\]
Elementary signals (cont.)

7. Unit ramp function

\[ r(t) = \begin{cases} 
  t, & t \geq 0 \\
  0, & t < 0 
\end{cases} \]

\[ r(t) = \int_{-\infty}^{t} u(\tau) d\tau = \int_{0}^{t} 1 d\tau = tu(t) \]

\[ r[n] = \begin{cases} 
  n, & n \geq 0 \\
  0, & n < 0 
\end{cases} = n \cdot u[n] \]

\[ \frac{dr(t)}{dt} = \begin{cases} 
  1, & t > 0 \\
  0, & t < 0 
\end{cases} \]

\[ f(t) \]

\[ p(t) = r(t-1) \]

\[ r(t) = 2r(t-1) + ? \]
System properties/characteristics

(a) 
\[ x(t) \xrightarrow{H} y(t) \]

(b) 
\[ x[n] \xrightarrow{H} y[n] \]

Notation: Let \( \mathcal{H} \) represent the system, \( x(t) \xrightarrow{\mathcal{H}} y(t) \) represent a system with input \( x(t) \) and output \( y(t) \).

1. Stability: BIBO (bounded input \( \Rightarrow \) bounded output) stability

\[ |x(t)| \leq M < \infty \]

\[ |y(t)| \leq N < \infty \]

System properties/characteristics (cont.)

E: Is the system with \( y(t) = e^{at} x(t) \) for \( a > 0 \) stable (BIBO)?

\[ y(t) = e^{at} \]

\( a = 0 \)

\( a < 0 \)

\( t = -\infty \)

\[ y[n] = \sum_{k=0}^{\infty} \rho^k x[n-k] \]

\[ y(t) = e^{at} x(ct), \ u(ct) \]
2. Memory/memoryless

- Memory system: present output value depends on future/past input.
- Memory/less system: present output value depends only on present input.

E: Memory systems:

\[
\begin{align*}
    y(t) &= 5x(t) + \int_{-\infty}^{t} x(\tau) d\tau \\
    y[n] &= \sum_{m=-5}^{n-1} x[m]
\end{align*}
\]

Memoryless systems:

\[
\begin{align*}
    y[n] &= x[n] + x^2[n]
\end{align*}
\]

3. Causal/noncausal

- Causal: present output depends on present/past values of input.
- Noncausal: present output depends on future values of input.

Note: Memoryless \(\Rightarrow\) causal, but causal not necessarily be memoryless.

4. Time invariance (TI): time delay or advance of input \(\Rightarrow\) an identical time shift in the output.

Let us define a system mapping \(y(t) = \mathcal{H}(x(t))\). The system is time-invariant if

\[
\begin{align*}
    x(t) &\xrightarrow{\mathcal{H}} y(t) \\
    x(t - t_0) &\xrightarrow{\mathcal{H}} y(t - t_0) & CT \\
    x[n - n_0] &\xrightarrow{\mathcal{H}} y[n - n_0] & DT
\end{align*}
\]
System properties/characteristics (cont.)

E: Is system \( y[n] = r^n x[n] \) time invariant?

\[
x[n] = X(n-n_0) \quad y[n] = x[n]. r^n
\]

\[
\circ \quad y[n-n_0] = r^n x[n-n_0] \quad \neq \quad y[n]
\]

- \( y(t) = e^{at} x^2(t) \):
  
  \[
  y(t) = e^{at} x(t) = e^{at} (x(t))^2
  \]

- \( y[n] = u[n] x[n] \):
  
  \[
  y(t) = u(t) x(t) \quad m(t-t_0) \neq y(t)
  \]

\[
\begin{align*}
\begin{cases}
  y(t) & t > 0 \\
  0 & t \leq 0
\end{cases}
\end{align*}
\]

\[
y(t) = u(t-\tau(t)) x(t)
\]\n
\[
y(t-\tau(t)) = u(t-\tau(t)) x(t-\tau(t))
\]
System properties/characteristics (cont.)

5. Linearity

Linear system: If \( x_1(t) \stackrel{\mathcal{H}}{\rightarrow} y_1(t) \), \( x_2(t) \stackrel{\mathcal{H}}{\rightarrow} y_2(t) \), then
\[
\begin{align*}
ax_1(t) + bx_2(t) & \stackrel{\mathcal{H}}{\rightarrow} ay_1(t) + by_2(t) \\
\end{align*}
\]
Else, nonlinear.

- Superposition property (addition)
- Homogeneity (scaling)

The following operations preserve linearity

- \( \frac{dx(t)}{dt} \stackrel{\mathcal{H}}{\rightarrow} \frac{dy(t)}{dt} \)
- \( \int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{H}}{\rightarrow} \int_{-\infty}^{t} y(\tau)d\tau \)
- \( \sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{H}}{\rightarrow} \sum_{m=-\infty}^{n} y[m] \)

System properties/characteristics (cont.)

E:

- \( y[n] = nx[n - 3] \): linear
  \[
  \begin{align*}
  x[n] & = a \cdot x[n] + b \cdot x[n-3] \\
  y'[n] & = n \cdot x[n-3] \\
  & = n \cdot (a \cdot x[n-3] + b \cdot x[n-3]) \\
  \end{align*}
  \]

- \( y(t) = 5x(t + t_0) \): linear
  \[
  \begin{align*}
  y[n] & = 5x[n] + 5x[n-3] \\
  y'[n] & = 5x[n-3] + 5x[n-3] \\
  \end{align*}
  \]

- \( y(t) = |x(t)| \): nonlinear
\[ y'[n] \leftrightarrow x'[n] \quad x'[n] \]

\[ x_1[n] = a \times x'[n] \]
\[ x_2[n] = b \times x'[n] \]

\[ y'[n] = a \times y[n] + b \times y[n] \]

\[ y[n] = n \times x[n-3] \]
\[ = n \left( a \times x[n-3] + b \times x[n-4] \right) \]