ECE351: Signals and Systems I

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Fundamentals of Signals and Systems

- Signal: a function of one or more variables (e.g., time, distance) that convey information on the nature of a physical phenomenon.
  - ⋆ Examples: heartbeat, blood pressure, temperature, vibration.
  - ⋆ One-dimensional signals: function depends on a singe variable, e.g., $x(t)$
  - ⋆ Multi-dimensional signals: function depends on two or more variables, e.g., video - time and two spatial dimensions

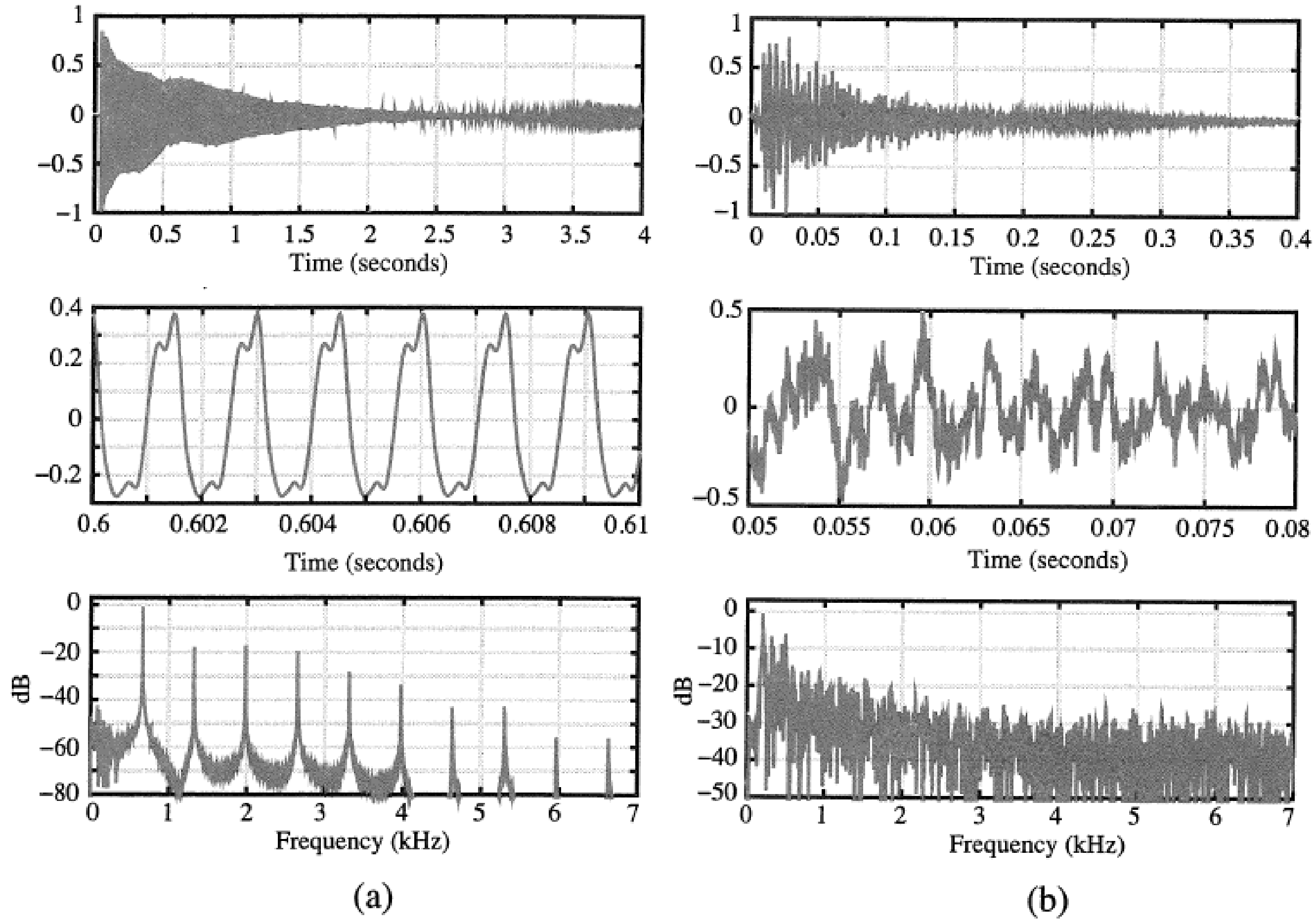


Figure 1.8 Waveforms of (a) a guitar and (b) a bass drum. (Courtesy of Maximilian Schäfer, University of Erlangen-Nürnberg, Erlangen, Germany.)

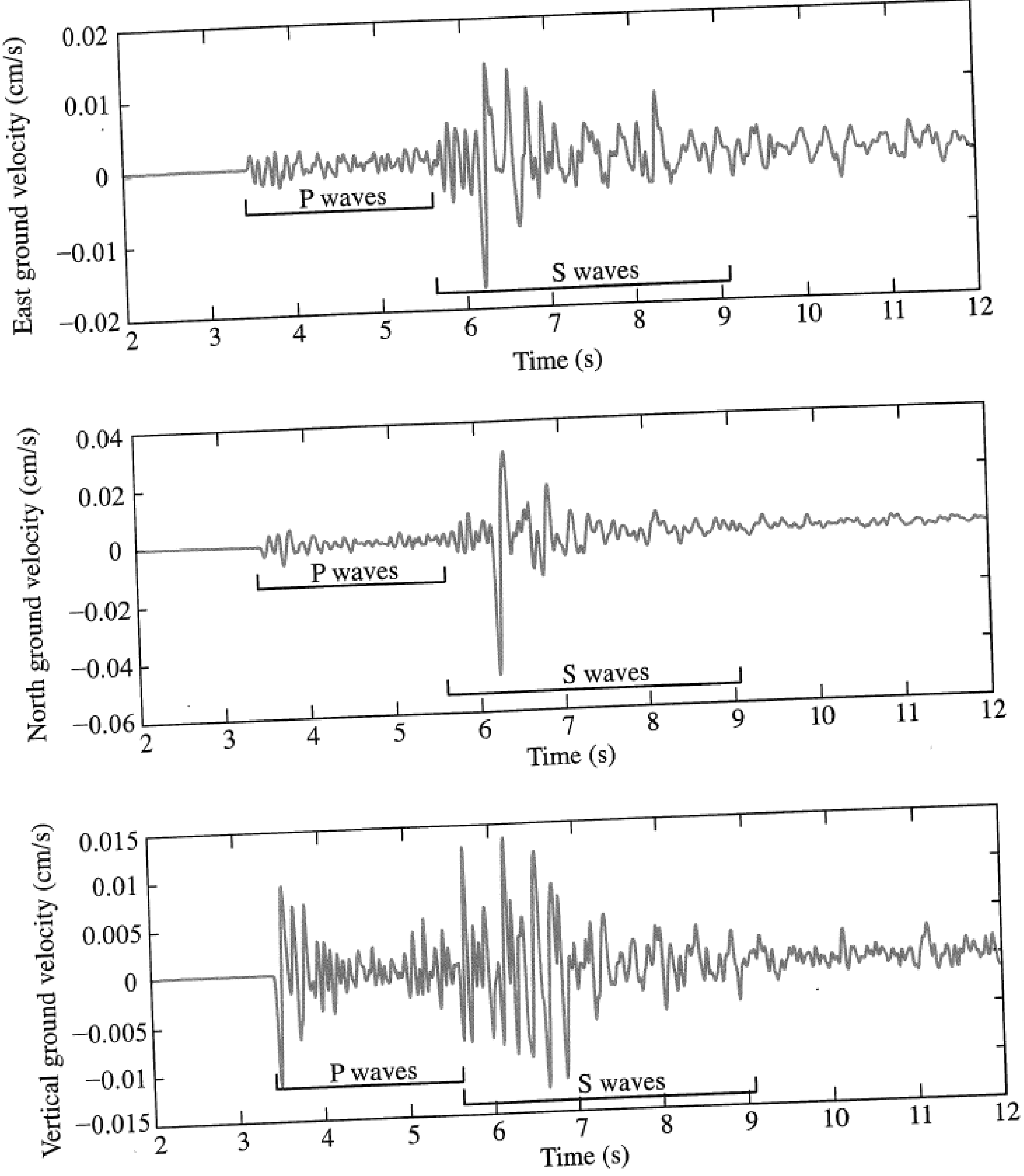


Figure 1.12 Tracings of the Chino Hills aftershock recorded at the Puddingstone Reservoir Station of the Southern California Seismic Network. Southern California Earthquake Data Center, 29 July 2008. Approximate durations of the P- and S-waves have been added to the original seismograph.

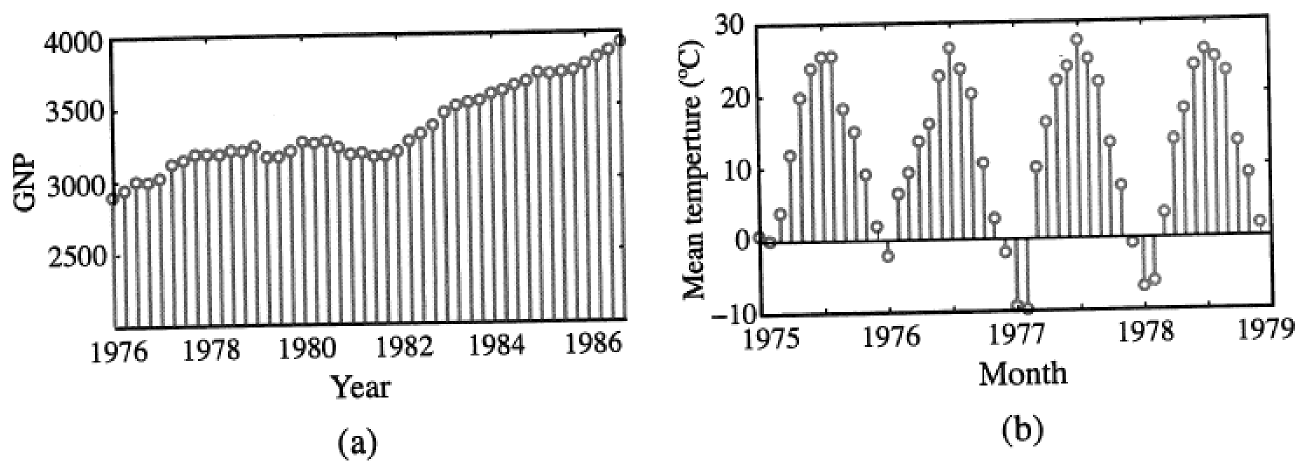


Figure 1.14 (a) Seasonally adjusted quarterly gross national product of the United States in 1982 dollars from 1976 to 1986. (Adapted from [Lüt91].) (b) Monthly mean temperature in degrees Celsius of St. Louis, Missouri, for the years 1975 to 1978. (Adapted from [Mar87].)

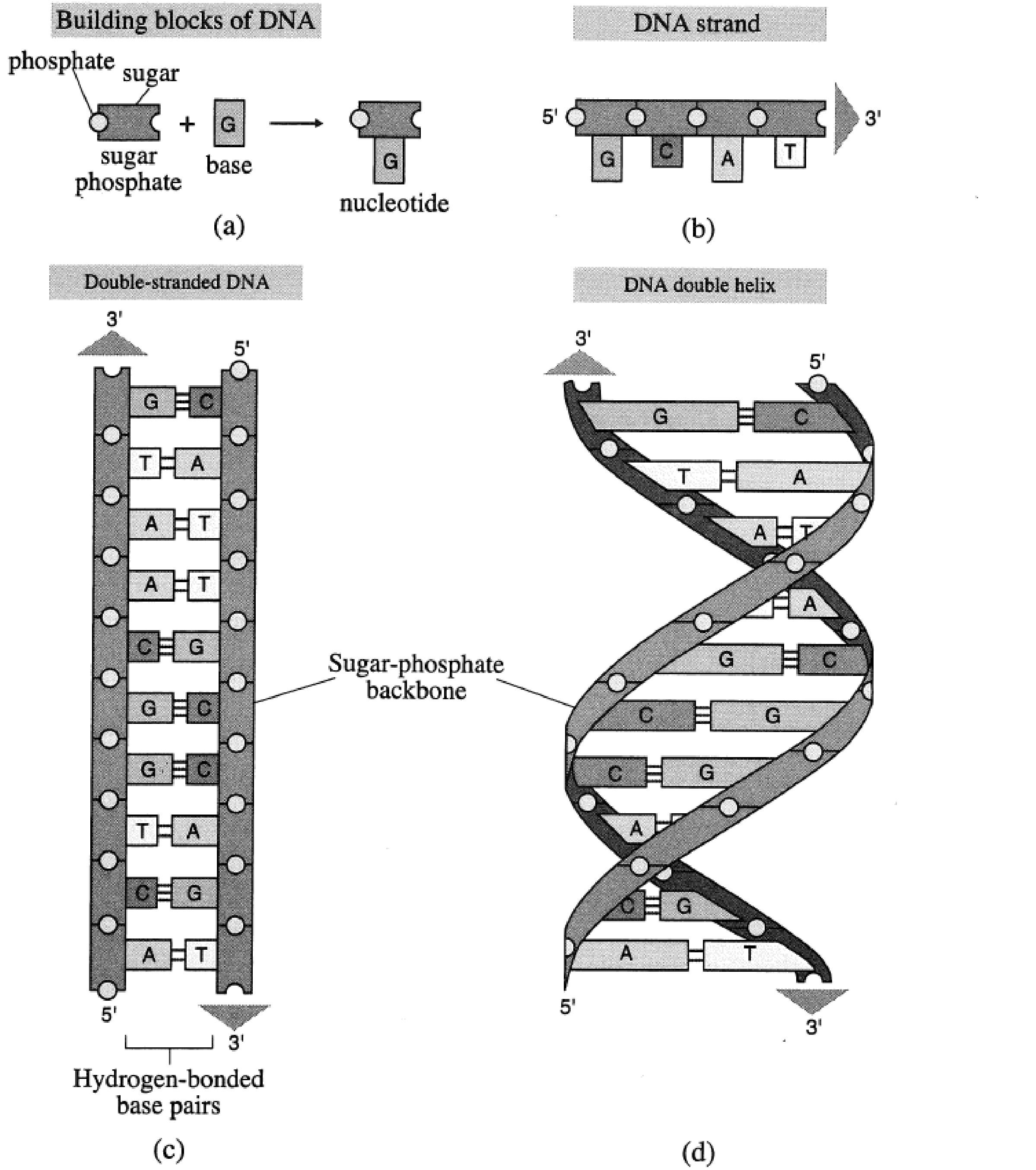


Figure 1.16 (a) Building block of DNA. (b) DNA strand. (c) Double-stranded DNA. (d) DNA double helix. (©1997 from "Essential Cell Biology," 1st edition, by Alberts et al. Reproduced by permission of Garland Science/Taylor & Francis Group, LLC.)

Fundamentals of Signals and Systems (cont.)

- System: an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.

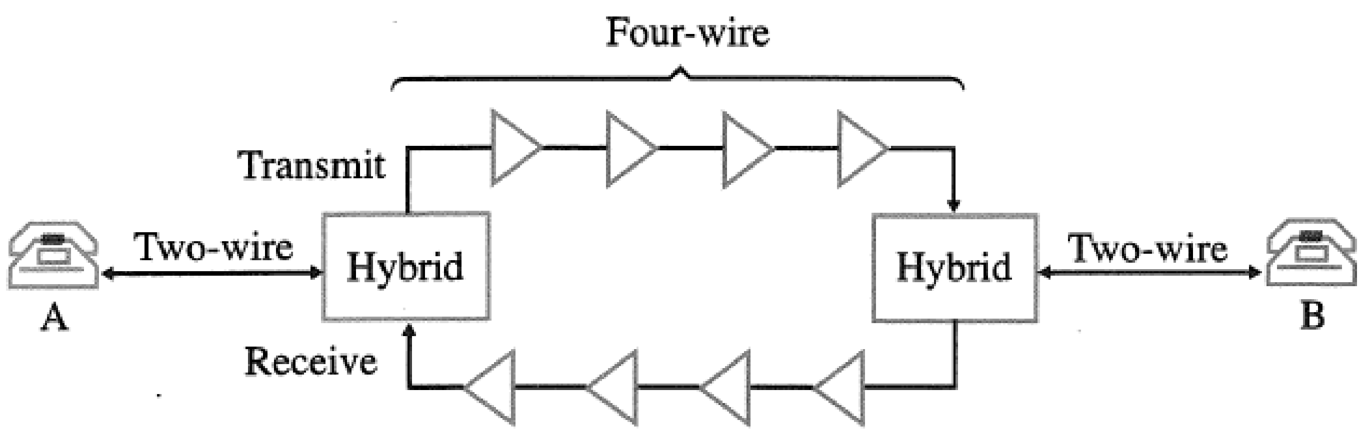


Figure 1.20 Basic 2/4-wire interconnection scheme.

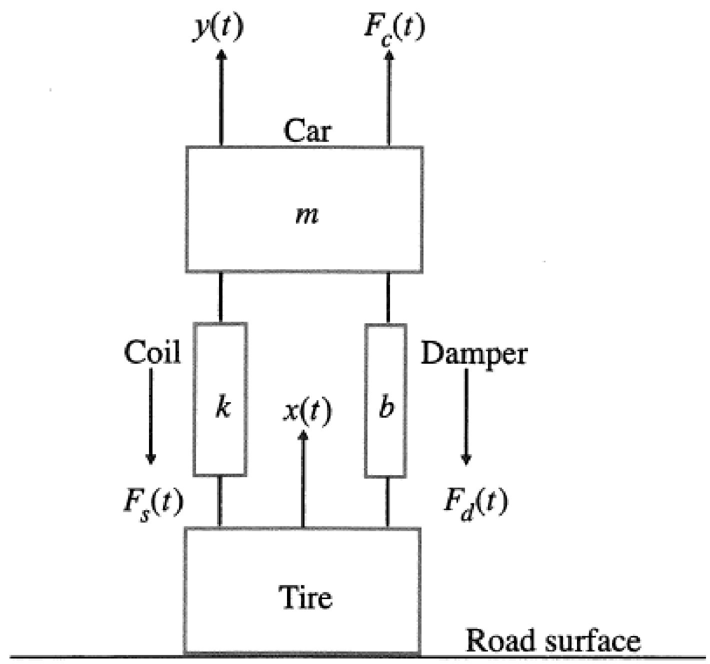


Figure 1.19 A free-body diagram of an automobile shock absorber system.

Fundamentals of Signals and Systems (cont.)

- Analog signal processing (ASP): use analog circuits such as resistors, capacitors, inductors, transistors, and diodes.
  - ⋆ Real time.
- Digital signal processing (DSP): adders, multiplers, memory.
  - ⋆ Flexible and repeatable.
- Notation:
  - ⋆ $x(t)$-Continuous time (CT) signals.
  - ⋆ $x[n]$-discrete time (DT) signals (n integers)

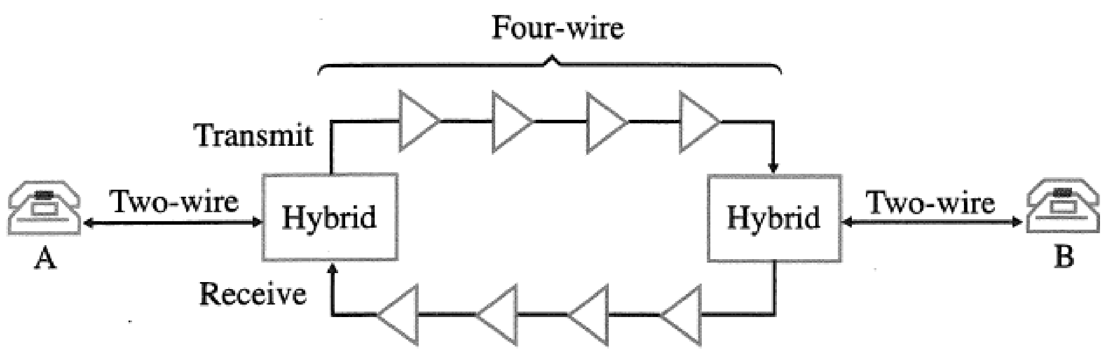


Figure 1.20 Basic 2/4-wire interconnection scheme.

Classification of signals

- Based on features:

1. CT and DT signals:

Classification of signals (cont.)

- For many cases, $x[n]$ is obtained by sampling $x(t)$ as:
  $x[n] = x(nT_s) = x(t)|_{t=nT_s},\ n = -\infty, \cdots, 0, \cdots, \infty$
- $x(t)$ must be recoverable from $x[n]$
- Are there any requirements for the sampling?

Classification of signals (cont.)

2. Even and odd signals:

Classification of signals (cont.)

- $x(t)$ is conjugate symmetric if $x(-t) = x^*(t)$, where $x(t) = a(t) + jb(t)$, $j = \sqrt{-1}$ and $a(t)$ and $b(t)$ are real.

$$\begin{cases} x^*(t) = a(t) - jb(t) \\ x(-t) = a(-t) + jb(-t) \end{cases}$$

- If $\Re\{x(t)\}$ is even and $\Im\{x(t)\}$ is odd, then $x(t)$ is conjugate symmetric.

Classification of signals (cont.)

3. Periodic and non-periodic signals:
   CT signal: if $x(t) = x(t + T_p), \ \forall t$, then $x(t)$ is periodic.

   - Fundamental period: T_p
   - Fundamental frequency $f_p = 1/T_p$ (Hz or cycles/second)
   - Angular frequency: $\omega_p = 2\pi f_p = 2\pi/T_p$ (rad/seconds)

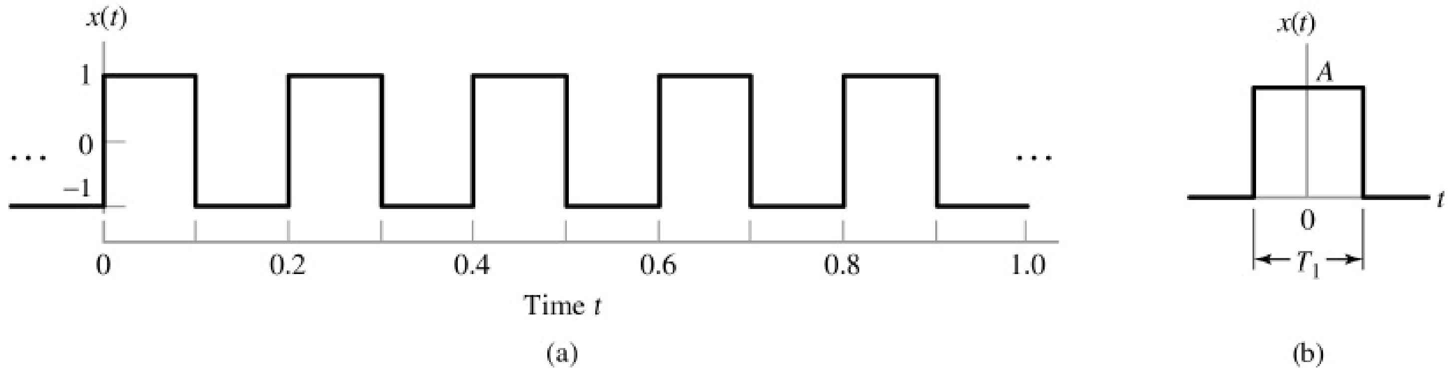
   DT signal: if $x[n] = x[n + N_p], \ \forall n$, then $x[n]$ is periodic.

   - N_p: $N_p > 0$ integer. min(N_p): fundamental period
   - N_p: samples/period, if the unit of n is designated as samples.
   - $F_p = 1/N_p$ (cycles/sample)
   - $\Omega_p = 2\pi F_p$ (rads/sample). If the unit of n is designated as dimensionless, then Ω_p is simply in radians.

   Note: A sampled CT periodic signal may not be DT periodic.
   Condition for DT signals to be periodic:

   - $x[n] = x(nT_s)$

Classification of signals (cont.)



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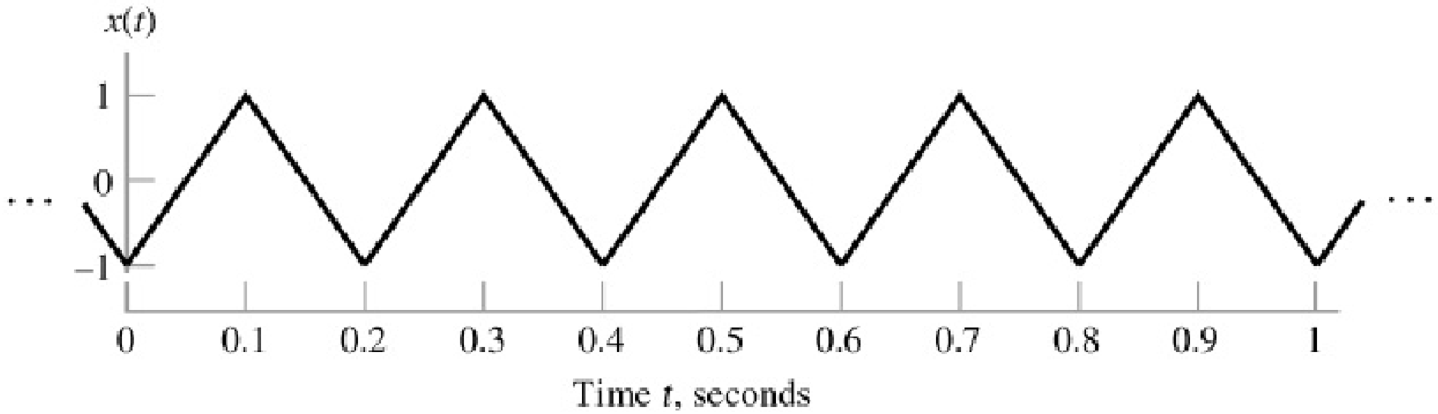

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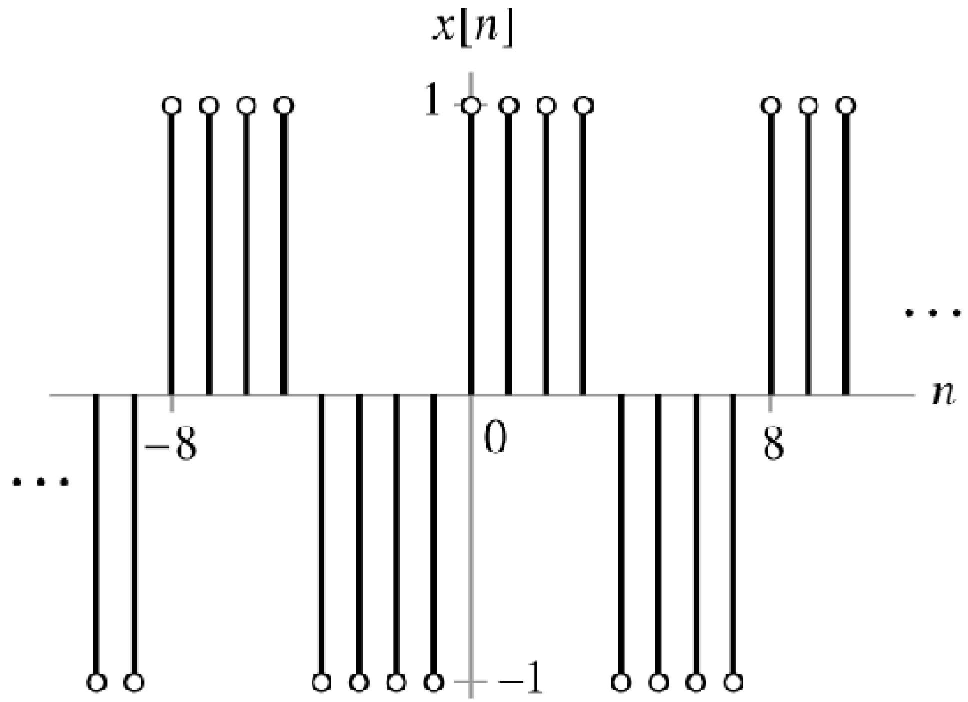
Classification of signals (cont.)



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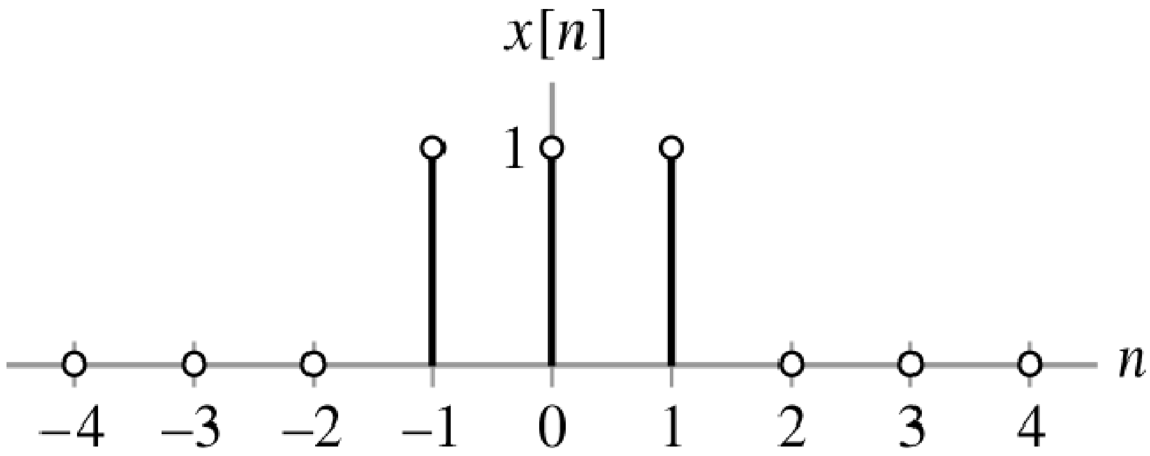


(d)

Classification of signals (cont.)

E: $x[n] = A\cos(2\pi F_p n + \theta)$. Condition for $x[n]$ to be periodic?

E: $x(t) = \sin^2(20\pi t)$ periodic? non-periodic?

Classification of signals (cont.)

- If $x(t)$ is sampled at $t = nT_s$ with $T_s = \frac{1}{4\pi} \longrightarrow$

- If $x(t)$ is sampled at $t = nT_s$ with $T_s = \frac{1}{40} \longrightarrow$

Classification of signals (cont.)

Sum of signals

CT signal:

If $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 and T_2. How about $x(t) = x_1(t) + x_2(t)$?

E: $x_1(t) = \cos(\pi t/2), x_2(t) = \cos(\pi t/3), x_1(t) + x_2(t)$

Classification of signals (cont.)

DT signal:

If $$\begin{cases} x_1[n] = x_1[n+N_1] \\ x_2[n] = x_2[n+N_2] \end{cases}$$

$$x_1[n] + x_2[n] = x_1[n+N_{sum}] + x_2[n+N_{sum}], \forall n$$

Classification of signals (cont.)

5. Energy and power signals:

   - CT signal $x(t)$:

$$\text{Energy: } E = \int_{-\infty}^{\infty} x^2(t)dt$$

$$\text{Power: } P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \quad - \quad \text{non-periodic signal}$$

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \quad - \quad \text{periodic signal}$$

Classification of signals (cont.)

- DT signal $x[n]$:

$$\text{Energy: } E = \sum_{n=-\infty}^{\infty} x^2[n]$$

$$\text{Power: } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n] \quad - \quad \text{non-periodic signal}$$

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \quad - \quad \text{periodic signal}$$

- Energy signal: iff $0 < E < \infty$
- Power signal: iff $0 < P < \infty$

Example:

Basic operations on signals

Operation	CT	DT	Note
Amplitude scaling	$y(t) = cx(t)$	$y[n] = cx[n]$	$c > 1$: gain $c < 1$: atten
Addition	$y(t) = x_1(t) + x_2(t)$	$y[n] = x_1[n] + x_2[n]$	
Multiplication	$y(t) = x_1(t)x_2(t)$	$y[n] = x_1[n]x_2[n]$	
Differentiation	$y(t) = \frac{d}{dt}x(t)$	(NO DT case)	
Integration	$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$	(NO DT case)	
Time scaling	$y(t) = x(at)$ $\begin{cases} a > 1: & \text{compression} \\ a < 1: & \text{expansion} \end{cases}$	$y[n] = x[kn]$ $k > 0$ and integer only	
Reflectio (time reversal)	$y(t) = x(-t)$	$y[n] = x[-n]$	
Time shifting	$y(t) = x(t - t_0)$ $\begin{cases} t_0 > 0: & \text{right shift} \\ t_0 < 0: & \text{left shift} \end{cases}$	$y[n] = x[n - n_0]$ $\begin{cases} n_0 > 0: & \text{right shift} \\ n_0 < 0: & \text{left shift} \end{cases}$	
Combination	$y(t) = x(at - t_0)$	$y[n] = x[kn - n_0]$	

Elementary signals

1. Exponential

CT	DT
$x(t) = Be^{at}$, a, B real	$x[n] = Br^n$
$\begin{cases} a < 0: & \text{decaying} \\ a > 0: & \text{growing} \\ a = 0: & \text{DC} \end{cases}$	$\begin{cases} 0 < r < 1: & \text{decaying} \\ r > 1: & \text{growing} \\ r = 1: & \text{DC} \end{cases}$

2. Sinusoidal

CT	DT
$x(t) = A\cos(\omega t + \phi)$	$x[n] = A\cos(\Omega n + \phi)$

Elementary signals (cont.)

Note:

- $x[n]$ May or may not be periodic
- If $\Omega N = 2\pi m$, m integer, or $\Omega = \frac{2\pi m}{N}$ (rads/sample), then $x[n]$ periodic: $x[n+N] = x[n]$
- Ω (unit?) rads/sample; N samples; ΩN radians (or simply radians if n is designated as dimensionless).

E: $x_1[n] = \sin\left(\frac{2\pi}{21}n\right)$, $x_2[n] = \sqrt{3}\cos\left(\frac{4\pi}{7}n\right)$. Fundamental period of $y[n] = x_1[n] + x_2[n]$?

Elementary signals (cont.)

3. Euler's identity

$$\begin{aligned} e^{j\theta} &= \cos(\theta) + j\sin\theta. \text{ Let } B = Ae^{j\phi} \\ Be^{j\omega t} &= Ae^{j\phi}e^{j\omega t} = Ae^{j(\omega t+\phi)} \\ &= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi) \end{aligned}$$

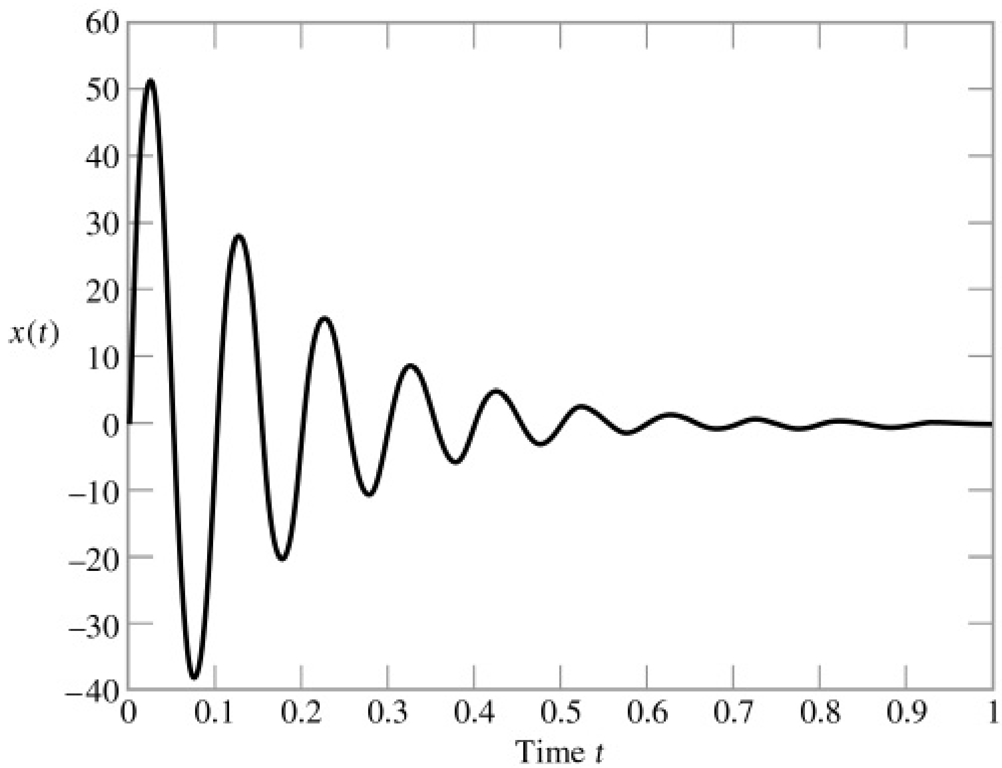
$$\begin{cases} A\cos(\omega t+\phi) = \Re\{Be^{j\omega t}\} \\ A\sin(\omega t+\phi) = \Im\{Be^{j\omega t}\} \end{cases}$$

Elementary signals (cont.)

4. Exponentially damped sinusoidal

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \phi),\ \alpha > 0 \text{ for damped}$$

$$x[n] = Br^{n}\sin(\Omega n + \phi),\ 0 < r < 1 \text{ for damped}$$



Elementary signals (cont.)

5. Step function

CT	DT
$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Note: $u(0)$ is not defined $u[0] = 1$.

Elementary signals (cont.)

E: Rectangular pulses in terms of $u(t)$ and $u[n]$.

$$x(t) = \begin{cases} A, & 0 \leq |t| \leq 0.5 \\ 0, & |t| > 0.5 \end{cases}$$

$x(t)$ can be expressed in terms of $u(t)$ as

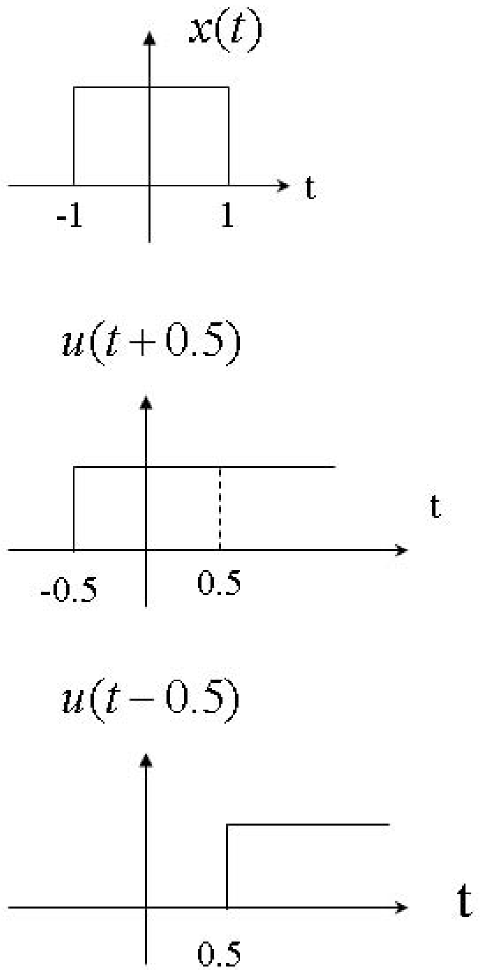
$$x(t) = Au(t + 1/2) - Au(t - 1/2)$$

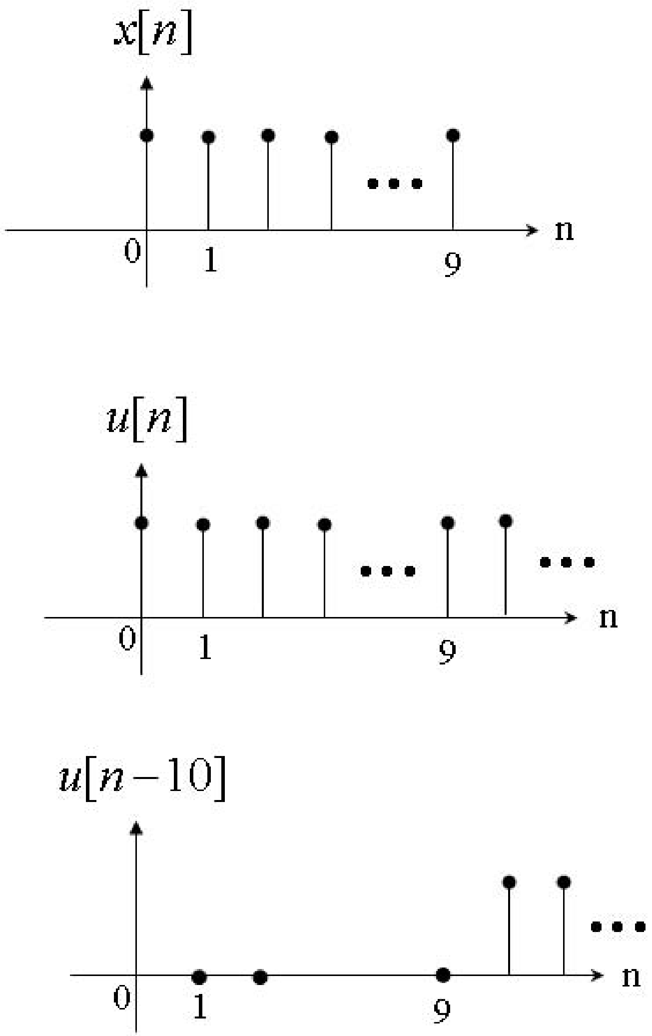
$$x[n] = \begin{cases} 1, & 0 \leq n \leq 9 \\ 0, & o.w. \end{cases}$$

$x[n]$ can be expressed in terms of $u[n]$ as

$$x[n] = u[n] - u[n - 10]$$

Elementary signals (cont.)





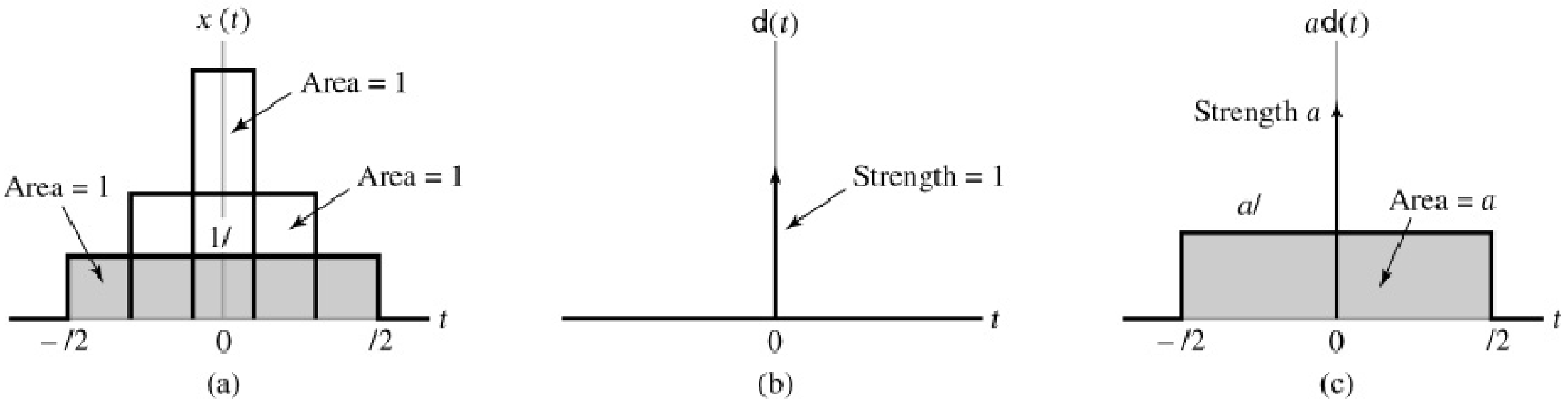
Elementary signals (cont.)

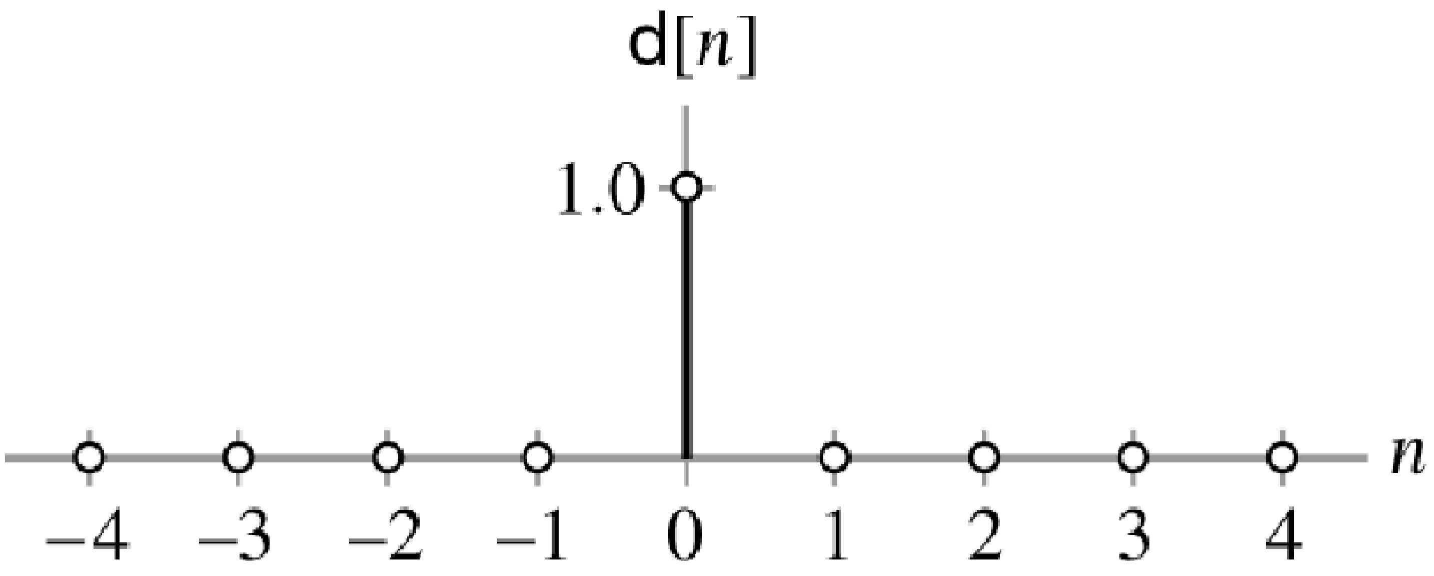
6. Impulse function

CT	DT
$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1 \end{cases}$	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

- Let $x_\Delta(t)$ be a rectangle with area one and width Δ and height *1/Δ*. Then, $\delta(t) = \lim_{\Delta \to 0} \Delta x_\Delta(t)$
- $\delta(-t) = \delta(t)$
- $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$
- $\delta(t) = \frac{d}{dt}u(t) \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
- $\delta(at) = \frac{1}{a}\delta(t), \ a > 0$
- $\int_{-\infty}^{\infty} x(t)\frac{d}{dt}\delta(t - t_0)dt = \frac{d}{dt}x(t)|_{t=t_0}$
- $\int_{-\infty}^{\infty} x(t)\frac{d^n}{dt^n}\delta(t - t_0)dt = \frac{d^n}{dt^n}x(t)|_{t=t_0}$

Elementary signals (cont.)





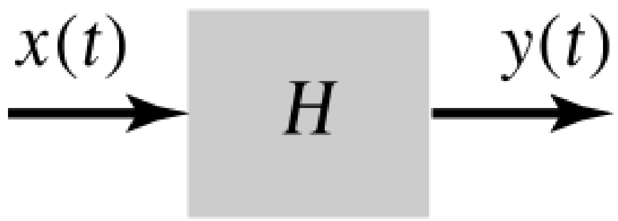
Elementary signals (cont.)

7. Unit ramp function

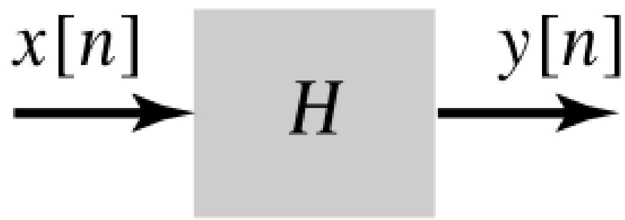
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} 1 d\tau = tu(t)$$

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

System properties/characteristics



(a)



(b)

Notation: Let $\mathcal{H}$ represent the system, $x(t) \xrightarrow{\mathcal{H}} y(t)$ represent a system with input $x(t)$ and output $y(t)$.

1. Stability: BIBO (bounded input $\Rightarrow$ bounded output) stability

System properties/characteristics (cont.)

E: Is the system with $y(t) = e^{at}x(t)$ for $a > 0$ stable (BIBO)?

$$y[n] = \sum_{k=0}^{\infty} \rho^k x[n-k]$$

System properties/characteristics (cont.)

2. Memory/memoryless

- Memory system: present output value depends on future/past input.
- Memoryless system: present output value depends only on present input.

E: Memory systems:

$$\begin{aligned} y(t) &= 5x(t) + \int_{-\infty}^{t} x(\tau) d\tau \\ y[n] &= \sum_{m=n-5}^{n+5} x[m] \end{aligned}$$

Memoryless systems:

$$y[n] = x[n] + x^2[n]$$

System properties/characteristics (cont.)

3. Causal/noncausal

   - Causal: present output depends on present/past values of input.
   - Noncausal: present output depends on future values of input.

   Note: Memoryless $\Rightarrow$ causal, but causal not necessarily be memoryless.

4. Time invariance (TI): time delay or advance of input $\Rightarrow$ an identical time shift in the output.
   Let us defin a system mapping $y(t) = \mathcal{H}(x(t))$. The system is time-invariant if

$$x(t - t_0) \xrightarrow{\mathcal{H}} y(t - t_0)$$

$$x[n - n_0] \xrightarrow{\mathcal{H}} y[n - n_0]$$

System properties/characteristics (cont.)

E: Is system $y[n] = r^n x[n]$ time invariant?

- $y(t) = e^{at} x^2(t)$:

- $y[n] = u[n] x[n]$:

System properties/characteristics (cont.)

5. Linearity

   Linear system: If $x_1(t) \xrightarrow{\mathcal{H}} y_1(t)$, $x_2(t) \xrightarrow{\mathcal{H}} y_2(t)$, then $ax_1(t) + bx_2(t) \xrightarrow{\mathcal{H}} ay_1(t) + by_2(t)$. Else, nonlinear.

   - Superposition property (addition)
   - Homogeneity (scaling)

   The following operations preserve linearity

   - $\frac{dx(t)}{dt} \xrightarrow{\mathcal{H}} \frac{dy(t)}{dt}$
   - $\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{H}} \int_{-\infty}^{t} y(\tau)d\tau$
   - $\sum_{m=-\infty}^{n} x[m] \xrightarrow{\mathcal{H}} \sum_{m=-\infty}^{n} y[m]$

System properties/characteristics (cont.)

E:

- $y[n] = nx[n-3]$: linear

- $y(t) = 5x(t+t_0)$: linear

- $y(t) = |x(t)|$: nonlinear

Basics of Matlab

1. Basic commands

   - ’help’: e.g., ’help elfun;’ ’help sinc;’ ’help square’
   - ’lookfor’: e.g., ’lookfor random;’ ’lookfor round;’ ’lookfor floor
   - ’plot’, ’stem’, ’rand’, ’randn’, ’sin’, ’cos’, ’exp’, ’sqrt’, ’zeros’, ’ones’, ’find’ ’xlabel’, ’ylabel’, ’title’, ’legend’
   - ’$\cdots$’: continue next line

2. Before a variable is applied, it must have a value.

   - $t = p$; will not work if p does not have a value. $p = sin(2 * pi * 0.3)$; $t = p$; works.
   - Sampling on time axis

$$
\begin{aligned}
t &= 0 : 0.001 : 1; \ \%\text{sampling interval} T_s = 0.001s \\
x &= \sin(2 * pi * t); \ \% \ x \text{ is a row vector. } '\sin(2 \ pi \ t)' \text{ will not work}
\end{aligned}
$$

Basics of Matlab (cont.)

3. Vector and matrix operations

- '$(\cdot)'$': complex conjugate transpose. E.g., $A = [1;\ 1{+}j0.7]$ is a 2×1 column vector. Then, $B = A'$ yields a 1×2 row vector as $B = [1\ \ 1{-}j0.7]$.
- '$(\cdot).'$': transpose.
- '$.*$': element-by-element multiplication. E.g., $A = [1\ \ 2;\ 3\ \ 4]$; $B = [4\ \ 5;\ 6\ \ 7]$; $C_1 = A.*B$; ($C_1 = [4\ \ 10;\ 18\ \ 28]$)
- '$*$': multiplication of matrices, vectors, and scalars. E.g, $C_2 = A*B$; ($C_2 = [16\ \ 19;\ 36\ \ 43]$).
- '$./$' and '$/$': element-by-element division and division, respectively. E.g., $A = [1\ \ 2\ \ 3]$; $B = [2\ \ 2\ \ 4]$; '$C = A/B$;' would NOT work. $C = A./B$; ($C = [0.2\ \ 1\ \ 0.75]$)
- '$(\cdot) \wedge n$': n-th power of a scalar. '$(\cdot). \wedge n$': n-th power of a vector or a matrix. E.g., $A = 1.4142$; $B = A \wedge 2$; ($B = 2$). $A = [1.4142\ \ 1.7321]$; '$B = A \wedge 2$;' would NOT work. $B = A. \wedge 2$; ($B = [2\ \ 3]$).

Time-Domain Representation of LTISystems

- System $\mathcal{H}$ is a linear time-invariant (LTI) system.
- How to analyze a system. Given an input, find system output.
- Impulse response of an LTI system $\mathcal{H}$:

Convolution sum

$$\begin{aligned} x(t) &= \delta(t) \xrightarrow{\mathcal{H}} y(t) = h(t) \\ x[n] &= \delta[n] \xrightarrow{\mathcal{H}} y[n] = h[n] \end{aligned}$$

where $h(t)$ (CT) and $h[n]$ (DT) are the system impulse responses.

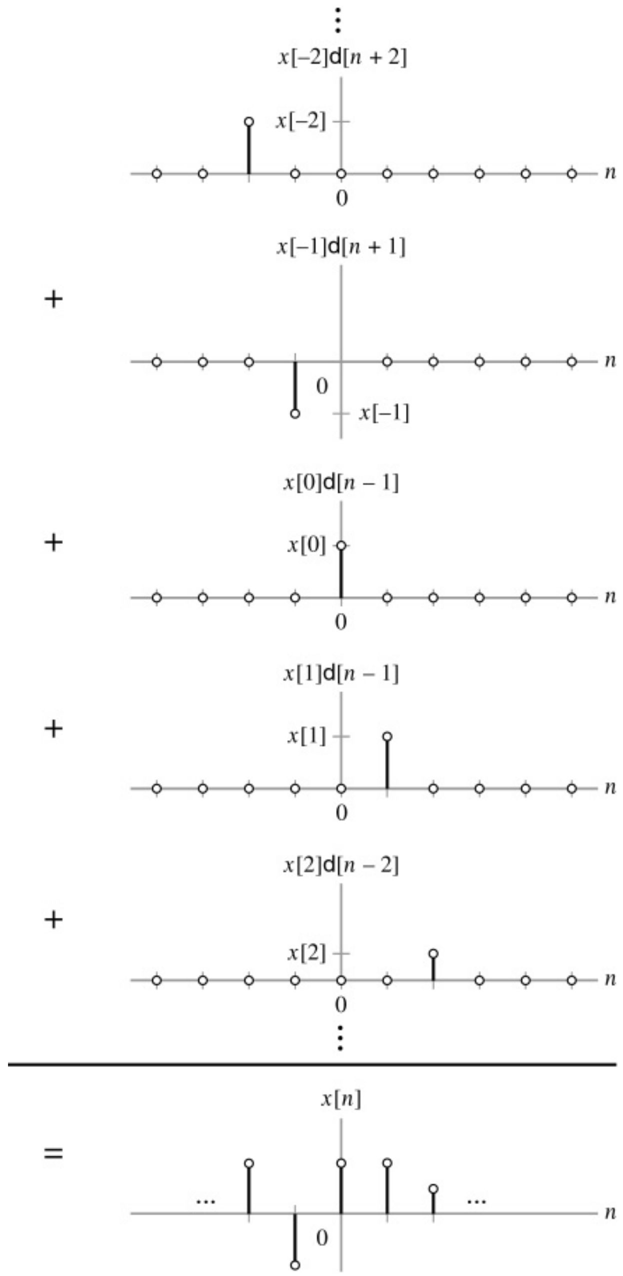
- $\star$ $h(t)$ or $h[n]$ completely characterizes an LTI system.
- $\star$ By knowing $h(t)$ or $h[n]$, system output can be obtained for an arbitrary input signal $x(t)$ or $x[n]$.
- $\star$ How is $y(t)/y[n]$ related to $x(t)/x[n]$ and $h(t)/h[n]$?

Convolution sum (cont.)

We will start with DT systems, and then analyze CT systems.

- Any signal $x[n]$ can be expressed as a sum of time-shifted impulses as (shown graphically next slide)

Convolution sum (cont.)



Convolution sum (cont.)

- Convolution sum: $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

- Properties of convolution

  ⋆ $x[n] * h[n] = h[n] * x[n]$ ▮
  ⋆ $\delta[n] * h[n] = h[n]$ ▮
  ⋆ $\delta[n-k] * h[n] = h[n-k]$

E: A system with input-output relationship as

$$y[n] = x[n] + (1/2)x[n-1]$$

a) System impulse response?
b) Find $y[n]$ for

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & o.w. \end{cases}$$

Convolution sum (cont.)

Convolution sum evaluation procedure

Let $w_n[k] = x[k]h[n-k]$. Then $y[n]$ is expressed as

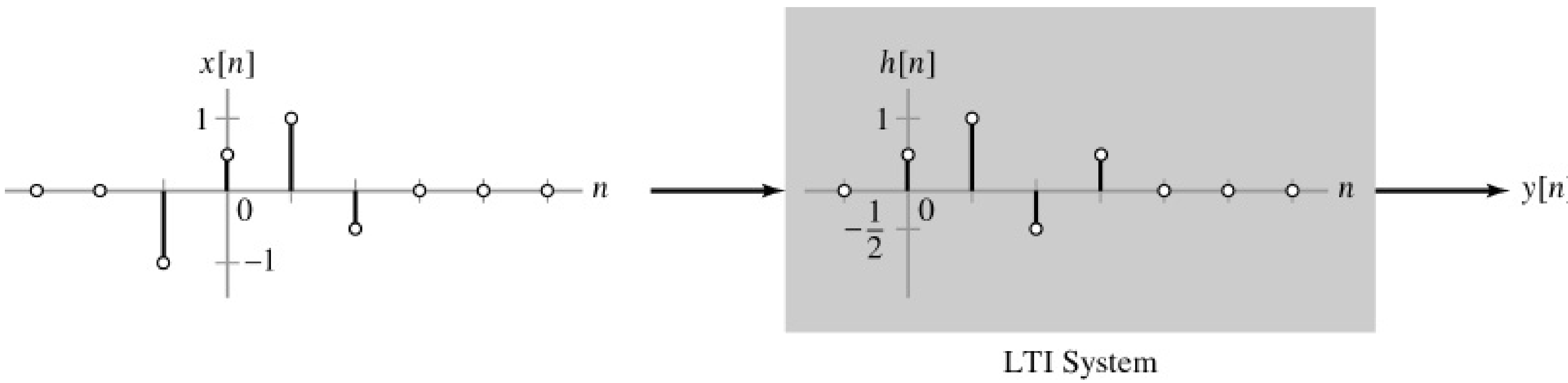
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} w_n[k]$$

1. Graph both $x[k]$ and $h[k]$
2. Time reversal $h[k] \longrightarrow h[-k]$
3. Time shift $h[-k]$ by n shifts $\longrightarrow h[n-k]$ (left shift)
4. For a specific n, form product $x[k]h[n-k]$
5. Sum all samples of $x[k]h[n-k] \longrightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution sum evaluation procedure (cont.)

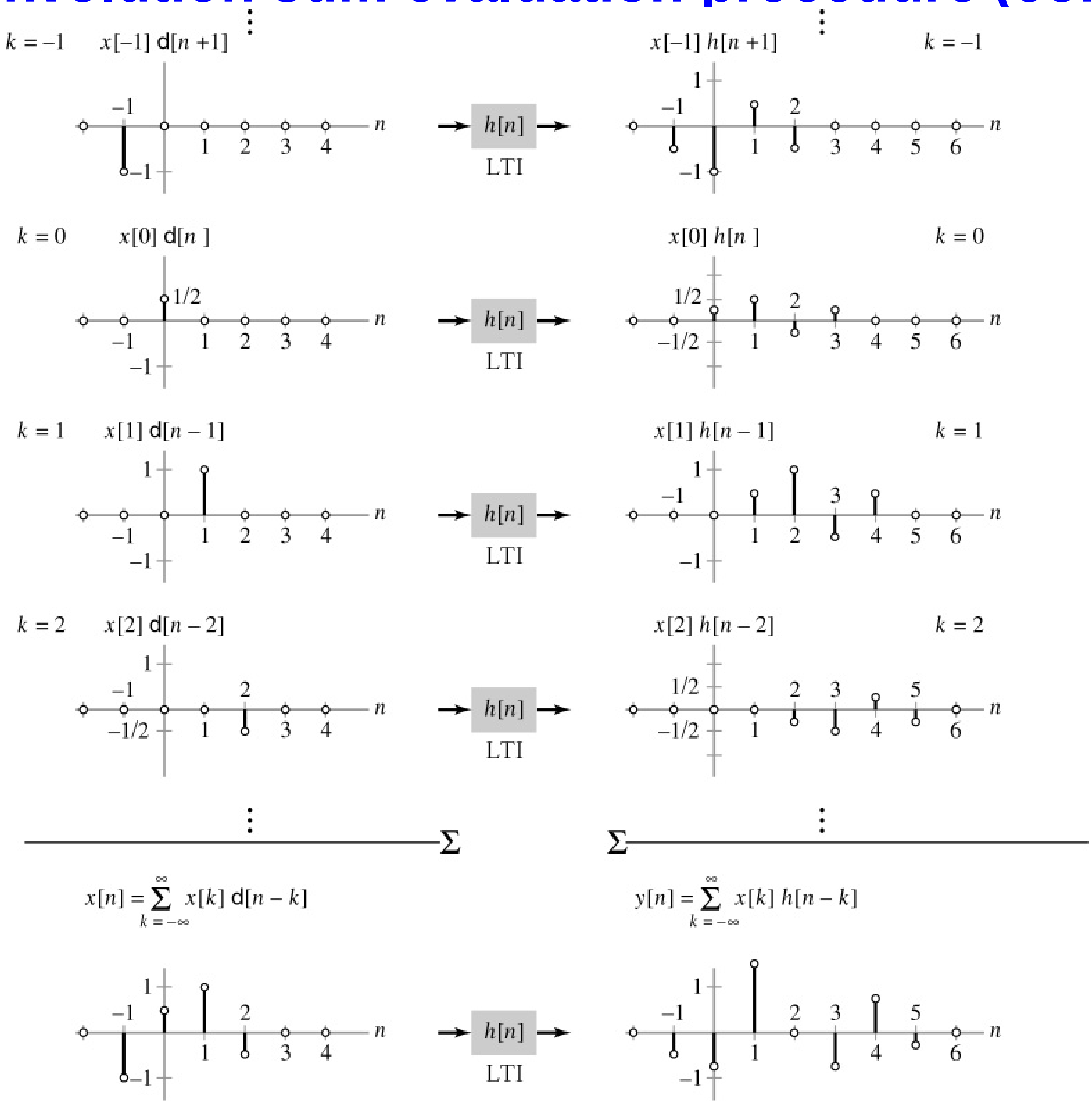
Graphical illustration of the convolution sum: (a) LTI system with impulse response $h[n]$ and input $x[n]$.



(a)

The decomposition of the input $x[n]$ into a weighted sum of time-shifted impulses results in an output $y[n]$ given by a weighted sum of time-shifted impulse responses next slide.

Convolution sum evaluation procedure (cont.)



Convolution sum evaluation procedure (cont.)

E: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ is applied to an LTI system with impulse response $h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$. Find $y[n]$.

$$y[n] = x[n] * h[n]$$

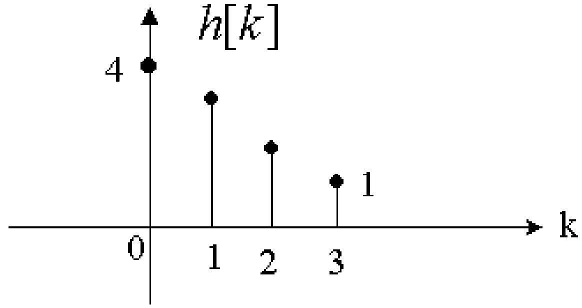
Exercise:

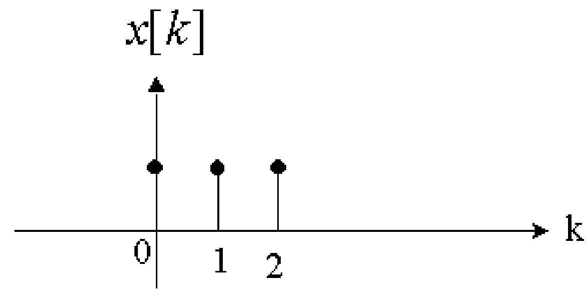
- Verify this the results using Matlab.

$$\begin{aligned} x &= [1 \;\; 1 \;\; 1]; \\ h &= [4 \;\; 3 \;\; 2 \;\; 1]; \\ y &= conv(x, h); \end{aligned}$$

Convolution sum evaluation procedure (cont.)

1)

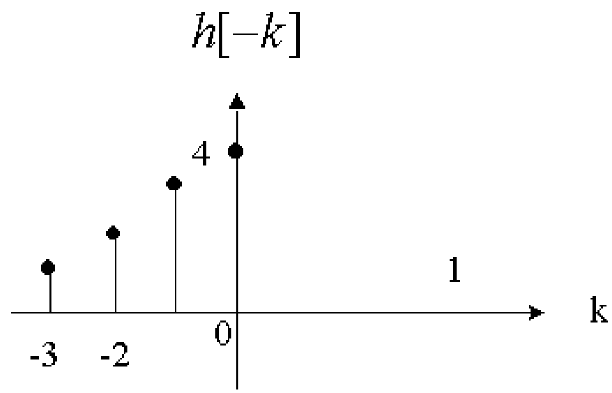




- Graph x[k] & h[k]

2) $h[-k]$

- Form h[-k] & time shift h[-k]



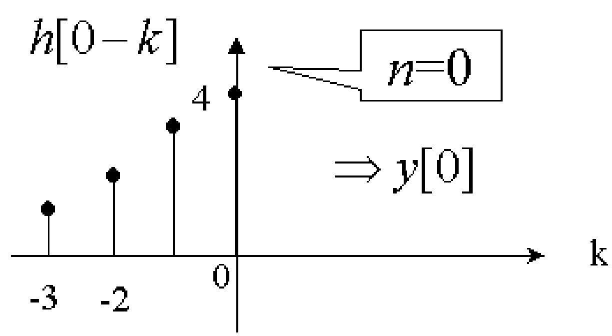
- For a specific *n*, form product

$$x[k]h[n-k] \Rightarrow \quad ...0 \quad 0 \quad 0 \quad 4 \quad 0 \quad ... \quad \Rightarrow y[0] = \sum \quad = 4$$

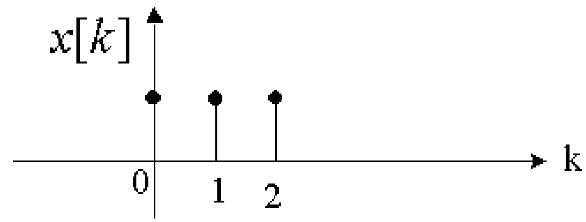
3) $h[n-k]$

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0 \quad \text{for } n < 0$$

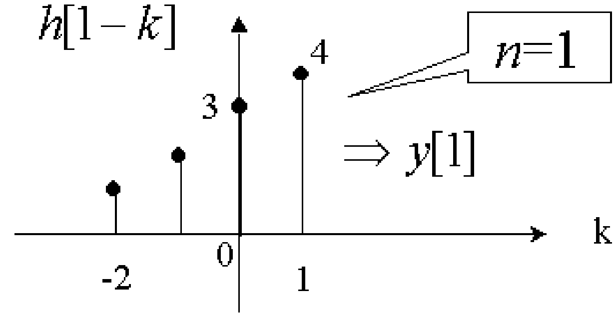
- Sum all products of $x[k]h[n-k]$



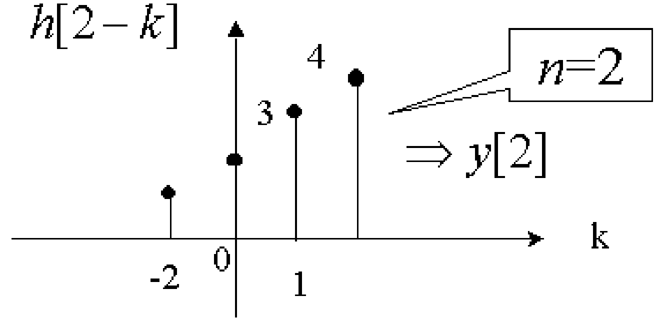
Convolution sum evaluation procedure (cont.)



$$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 3 \quad 4 \quad 0 \quad 0 \ldots \Rightarrow y[1] = \sum \quad = 7$$



$$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 2 \quad 3 \quad 4 \quad 0 \ldots \Rightarrow y[2] = \sum \quad = 9$$



$y[4] = 3$

$y[5] = 1$

$y[n] = 0 \text{ for } n \geq 6$

$$\ldots 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 0 \ldots \Rightarrow y[3] = \sum \quad = 6$$

Convolution integral

- For CT case.
- Recall DT case:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Note: Weighed SUM of time-shifted impulses. Similarly,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Note: Weighted superposition of time-shifted impulses.

$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

Convolution integral (cont.)

$$
\begin{aligned}
y(t) &= \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \quad \text{linear operators} \\
&= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\left\{\delta(t-\tau)\right\}d\tau
\end{aligned}
$$

$$
\delta(t-\tau) \xrightarrow{\mathcal{H}} h(t-\tau)
$$

Thus,

$$
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
$$

Note:

- ⋆ $x(t) * h(t) = h(t) * x(t)$
- ⋆ $\delta(t) * h(t) = h(t)$
- ⋆ $\delta(t-t_0) * h(t) = h(t-t_0)$

Convolution integral evaluation procedure

1. Graph $x(t)$ and $h(t)$

2. Time reverse $h(\tau) \Rightarrow h(-\tau)$

3. Time shift $h(-\tau)$ by $t \Rightarrow h(t-\tau)$

4. For a specific value of t, form product $x(\tau)h(t-\tau)$

5. Integrate $x(\tau)h(t-\tau) \Longrightarrow$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution integral evaluation procedure (cont.)

Example:

$$x(t) = u(t-1) - u(t-3) \xrightarrow{h(t)=u(t)-u(t-2)} y(t) = ??$$

Convolution integral evaluation procedure (cont.)

E: RADAR range measurement: RADAR-Radio Detection And Ranging:

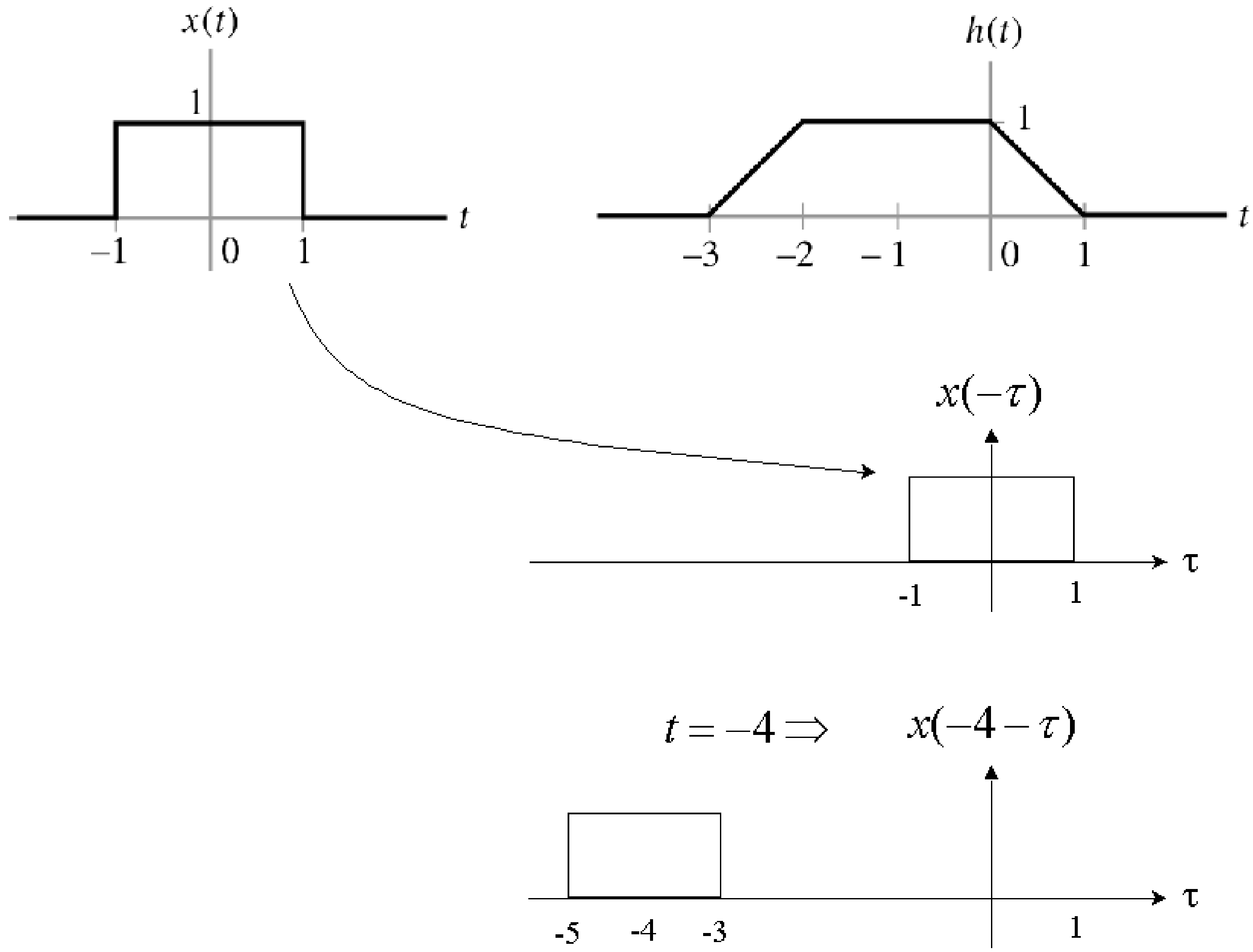
Tx: $$x(t) = \begin{cases} \sin(w_c t), & 0 \le t \le T_0 \\ 0, & o.w. \end{cases}$$

Typically,

$$h(t) = \alpha\delta(t-\beta),\ \beta > 0$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution integral evaluation procedure (cont.)



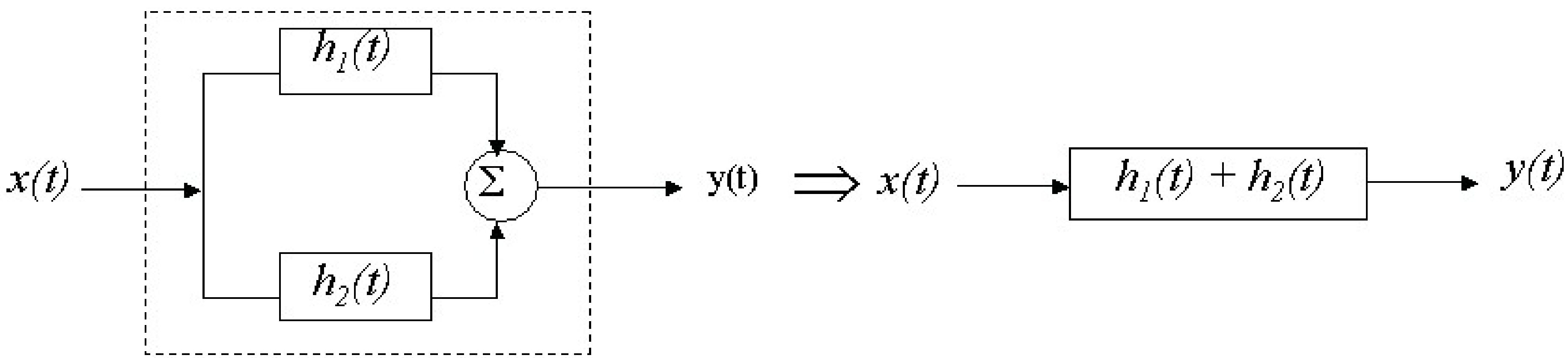
Interconnection of LTI systems

Given:

$$\left.\begin{array}{cc} h_1(t) & \xrightarrow{\mathcal{H}_1} \\ \vdots & \vdots \\ h_N(t) & \xrightarrow{\mathcal{H}_N} \end{array}\right\} \Longrightarrow \text{ form a bigger system } \xrightarrow{\mathcal{H}}$$

Question: How is $h(t)$ related to $h_1(t) \cdots h_N(t)$?

- Parallel Connection

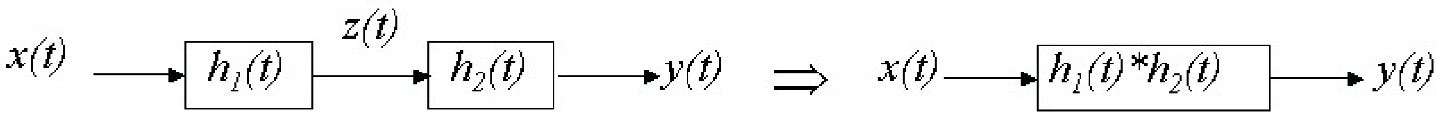


Interconnection of LTI systems (cont.)

- Distribution property of convolution process:

Interconnection of LTI systems (cont.)

- Cascade Connection



Interconnection of LTI systems (cont.)

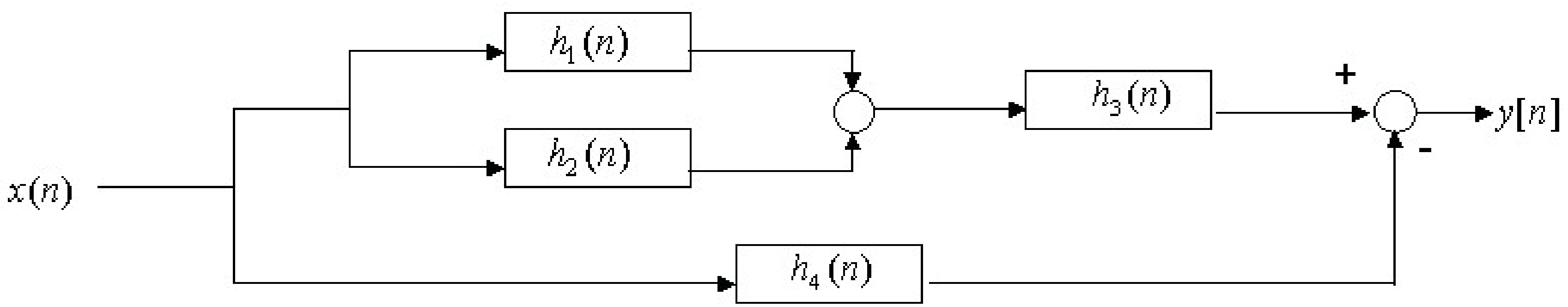
Let $\eta = \tau - \nu,\ d\eta = d\tau$ (for fixed ν). Then,

Interconnection of LTI systems (cont.)

- Associative Property (Same for DT)

- Commutative Property (Same for DT

E: Example 2.11, p_{130}: See figure below. Find the impulse response $h[n]$ of the overall system.

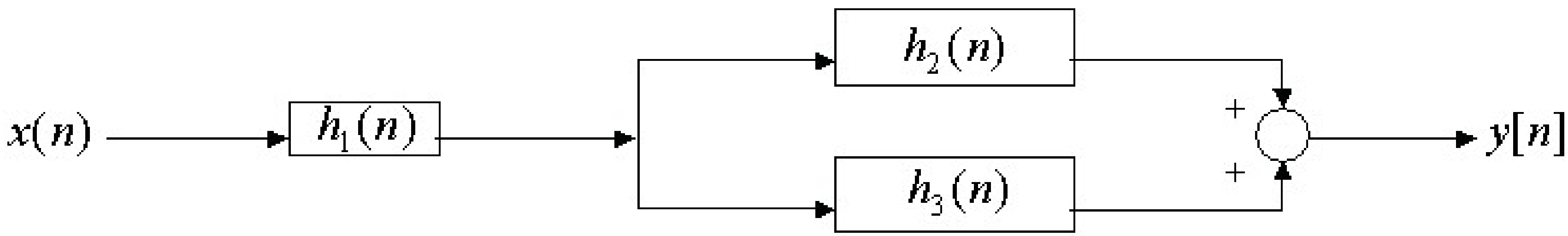


Interconnection of LTI systems (cont.)

$$\begin{cases} h_1[n] = u[n] \\ h_2[n] = u[n+2] - u[n] \\ h_3[n] = \delta[n-2] \\ h_4[n] = \alpha^n u[n] \end{cases}$$

Interconnection of LTI systems (cont.)

E: An interconnection of LTI system is depicted in the figure below. $h_1[n] = \left(\frac{1}{2}\right)^n u[n+2]$, $h_2[n] = \delta[n]$, and $h_3[n] = u[n-1]$. Find the impulse response $h[n]$ of the overall system.



LTI SYSTEM PROPERTIES & IMPULSE RESPONSE

System properties (Chapter1)

- Stability (BIBO)
- Memory (depend on current input only)
- Causality (does not depend on future inputs)
- Linearity
- Time invariance

Linearity + Time invariance → LTI

Memory + Causality + LTI → Memoryless, LTI

Stability + Memory + Causality + LTI → Memoryless, Stable, LTI

For LTI systems:

$h(t)$

$h[n]$ $\xrightarrow{\text{Completely determine}}$ Input-output behaviors

Thus, stability, memory, causality are related to h(t)/h[n].

a). If an LTI system is MEMORYLESS $\overset{iff}{\Longleftrightarrow}$

$$h[k] = c\delta[k]$$

$$h(\tau) = c\delta(\tau)$$

Proof:

b) If an LTI system is CAUSAL: $\Longleftrightarrow$ $\begin{array}{l} \text{DT}: h[k] = 0 \text{ for } k < 0 \\ \text{CT}: h(\tau) = 0 \text{ for } \tau < 0 \end{array}$

Proof:

c) If an LTI system is BIBO STABLE: $\Longleftrightarrow$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Proof:

E First-order autoregressive system

$$y[n] = \rho y[n-1] + x[n], \text{ with } h[n] = 0 \text{ for } n < 0$$

<u>STEP RESPONSE:</u>

LTI

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$
$$x(t) \qquad\qquad y(t)$$

Step response: if $x[n] = u[n] \Rightarrow y[n] = s[n] = \sum_{k=-\infty}^{n} h[k]$

Proof:

Example

$$u[n] \rightarrow \boxed{h[n] = \rho^n u[n]} \rightarrow y[n] = ?? \qquad |\rho| < 1$$

DIFFERENTIAL & DIFFERENCE EQUATION REPRESENTATIONS OF LTI SYSTEMS

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

$$x[n] \qquad\qquad y[n]$$

Input-output relation can be described as

CT
$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Linear constant-coefficient differential equation.

DT
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

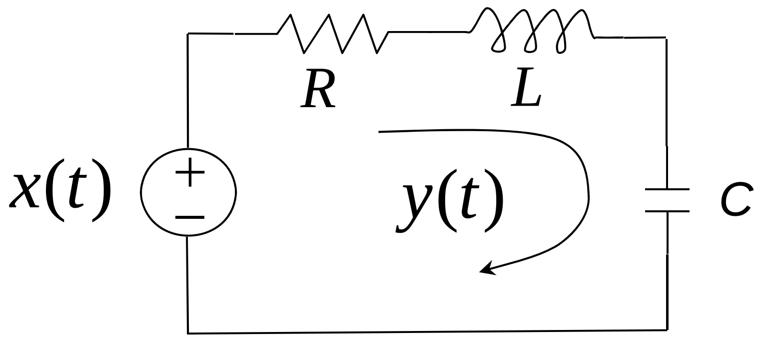
Linear constant-coefficient difference equation.

$a_{k,} b_k$ constants

- Order of differential/difference equation: *(N,M)*
- Often n ≥ M, and <u>order is described using *N* only</u>

E

Describe the following RLC circuit by a differential equation.



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{c}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

differentiates both sides

$$\frac{1}{c}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t)$$

E

2nd-order difference equation

$$y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n] + 2x[n-1]$$

$$\begin{cases} (a_0, a_1, a_2) = (1,\ 1,\ 1/4) \\ (b_0, b_1) = (1\ \ ,2) \end{cases}$$

y[n] can be evaluated recursively:

Need *y[-2], y[-1]:* Initial conditions. *x[-1]* depends on input applied.

For example, if $x[n] = \left(\frac{1}{2}\right)^n u[n], y[-1] = 1, y[-2] = -2$, then

$$y[0] = 1 + 2 \times 0 - 1 - \frac{1}{4} \times (-2) = \frac{1}{2}$$

$$y[1] = \frac{1}{2} + 2 \times 1 - \frac{1}{2} - \frac{1}{4} \times (1) = 1\frac{3}{4}$$

$$\vdots$$

SOLVING DIFFERENTIAL AND DIFFEENCE EQUATIONS

Just a review, rather than in depth:

Complete Solution

Solution forms:	Homogeneous		Particular
CT	$y(t) = y^{(h)}(t)$	$+$	$y^{(p)}(t)$
DT	$y[n] = y^{(h)}[n]$	$+$	$y^{(p)}[n]$
	$\Uparrow$		$\Uparrow$
	Natural response		Forced response
	Input signal $x(t)$ or $x[n] = 0$		Initial rest
	Depends on initial condition!		Depends on input signal!

GENERAL CASE

CT system: $y^{(h)}(t)$ is the solution of the *homogeneous* equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0$$

The homogeneous *solution* is of the form: $y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t}$

where r_i are the N roots of the system's *characteristic* equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

Note: c_i: to be determined (later) so that the complete solutions satisfy the initial conditions.

DT system: $y^{(h)}[n]$ is the solution of the homogeneous equation:

$$\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0$$

The homogeneous *solution* is of the form: $y^{(h)}[n] = \sum_{i=1}^{N} c_i r_i^n$

where r_i are the N roots of the system's *characteristic* equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

Note:

- c_i: same as CT case.
- CT and DT characteristic equations are different.

Example

- Homogeneous equation: $y[n] - \frac{1}{4} y[n-1] = 0$ (set *x[n]*=0)
- Characteristic equation: $\sum_{k=0}^{N} a_k r^{N-k} = 0$
- Solution of homogeneous equation: $y^{(h)}[n] = c_1 (\frac{1}{4})^n$

c_1: to be determined so that the complete solutions satisfy the initial conditions.

- A particular solution is assumed independent of the homogeneous solution

- Usually obtained assuming that output has the same form as input signals.

- The form of the particular solution associated with common inputs are summarized in the following table.

Continuous time		Discrete time	
Input	Particular solution	Input	Particular solution
1	c	1	c
t	c_1t+c_2	n	c_1n+c_2
e^{-at}	$c\ e^{-at}$	α^n	$c\ \alpha^n$
$\cos(\omega t+\phi)$	$c_1\cos(\omega t+\phi)+c_2\sin(\omega t+\phi)$	$\cos(\Omega n+\phi)$	$c_1\cos(\Omega n+\phi)+c_2\sin(\Omega n+\phi)$

INPUT~ PARTICULAR FORM

Example

Assume a particular solution of the form

$$y^{(p)}[n] = c_p (\frac{1}{2})^n$$ (for input in the form of $\alpha^n u[n]$)

Complete solution:

$$y[n] = y^{(h)}[n] + y^{(p)}[n] = 2(\frac{1}{2})^n + c_1(\frac{1}{4})^n \quad \text{for} \quad n \geq 0$$

c_1 is obtained from the initial condition.

$$y[0] = 3 \Rightarrow 3 = 2 + c_1 \Rightarrow \boxed{c_1 = 1}$$

$$\boxed{y[n] = 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^n,} \quad \text{for } n \geq 0$$

For a general 1st-order difference equation given as

$$y[n] + ay[n-1] = bx[n] \qquad n \geq 0$$

Assuming initial condition y[-1] and causal x[n] $(x[n] = 0 \text{ for } n < 0)$

* Solution:

$$y[n] = \underbrace{(-a)^{n+1} y[-1]}_{\text{natural response}} + \underbrace{\sum_{i=0}^{n} (-a)^{n-i} bx[i]}_{\text{forced response}}$$

- CT Case (also 1st-order only)

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

** Solution:

$$y(t) = \underbrace{e^{-at} y(0)}_{\text{natural response}} + \underbrace{\int_0^t e^{-a(t-\tau)} bx(\tau) d\tau}_{\text{forced response}} \qquad t \geq 0$$

(x(t) causal)

(Compare * and **!)

FOURIER REPRESENTATION OF SIGNALS & LTI SYSTEMS

CT: f cycle/second (Hz) $\omega = 2\pi f$ rads/s

DT: F cycles/sample $\Omega = 2\pi F$ rads/sample

- Basic signals as weighted superposition of impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xrightarrow[h[n]]{LTI} y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

superposition, weight ($x[k]$), delay ($\delta[n-k]$)

(LTI property)

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow[h(t)]{LTI} y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Time-domain waveform represents how fast signal changes.
  Signals in terms different frequency components or
  weighted superpositions of complex sinusoids.

$$\begin{cases} \text{CT: } X(f) \text{ or } X(\omega) \\ \text{DT: } X[k] \end{cases}$$

Why signals represented as weighted superpositions of complex sinusoids?

DT: $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$ ~ related to h[k]

Note:

a). $H(e^{j\Omega})$ is NOT a function of *n*, only a function of Ω.

$H(e^{j\Omega})$ Is called the frequency response.

b). System modifies the amplitude of input by $\left|H(e^{j\Omega})\right|$.

$\left|H(e^{j\Omega})\right|$: magnitude response.

c). System introduces a phase lag $\angle H(e^{j\Omega})$.

(the book uses $\arg\{H(e^{j\Omega})\}$)

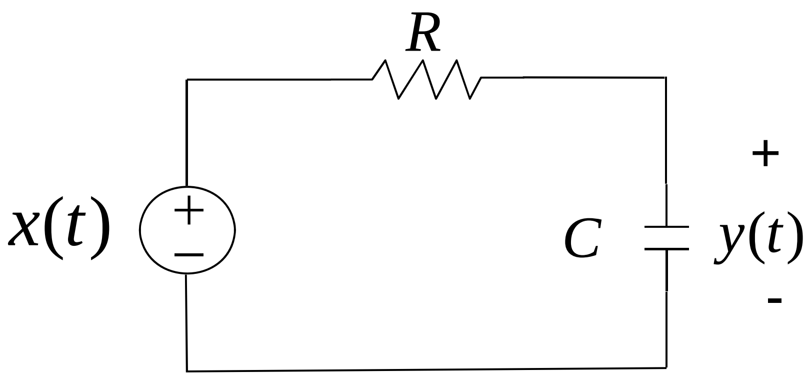
$$H(e^{j\Omega}) = \left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})}$$

CT: $$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

with $x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega)e^{j\omega t}$

$$= \left|H(j\omega)\right| e^{j(\omega t + \angle H(e^{j\Omega}))}$$

E Example



Impulse response: $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$

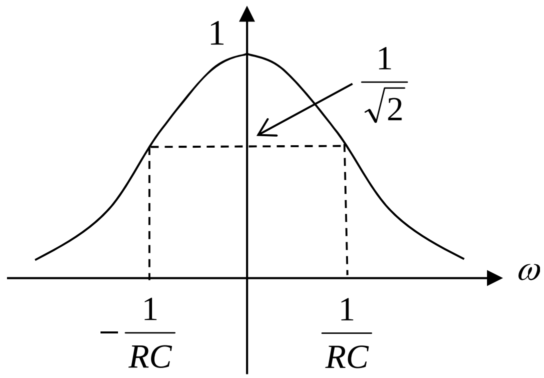
Find frequency response.

Solution:

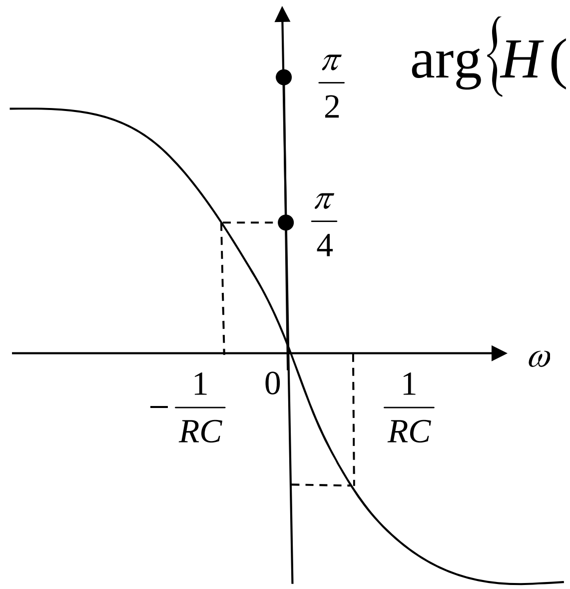
- Magnitude response:

- Phase response:

$$\left|H(j\omega)\right|$$



$$\arg\left\{H(e^{j\Omega})\right\} = \underline{\angle H(j\omega)}$$



$e^{j\omega t}$: eigenfunction of the LTI system (eigen value $\lambda = H(j\omega)$)

$$(H\left\{e^{j\omega t}\right\} = \lambda e^{j\omega t})$$

Now, if the input to an LTI system is expressed as a weighted sum of <u>*M* complex sinusoids</u>:

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}, \quad \text{then}$$

$$y(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$$

Fourier representations of four classes of signals

Time property	Periodic	Nonperiodic
Continuous time (t)	• Fourier Series (FS) $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T}$ $X[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt,$ (T: period)	• Fourier Transform (FT) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Discrete time [n]	• Discrete-Time Fourier Series (DTFS) $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}, \Omega_0 = \frac{2\pi}{N}$ $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$ (T: period)	• Discrete-Time Fourier Transform (DTFT) $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

<u>DTFS</u>: *x[n]* periodic with period *N*, fundamental freq. $\Omega_0 = 2\pi/N$
DTFS coefficients of *x[n]: X[k].* Then

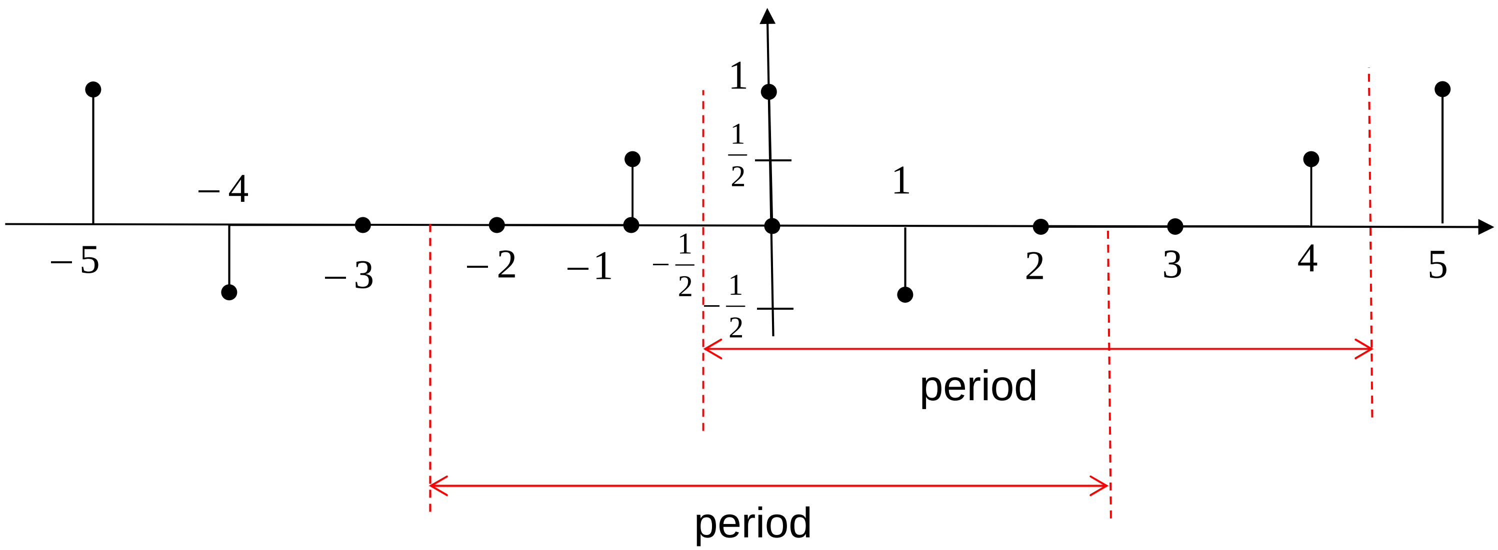
$$\begin{cases} x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \\ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} \end{cases}$$

Freq-domain representation of x[n]

x[n] and X[k] are a DTFS pair: $x[n] \xleftrightarrow{DTFS;\ \Omega_0} X[k]$

Note: a). Either x[n] or X[k] provides a complete description of the signal.
b). The limits on sums of x[n] or X[k] may be chosen differently from 0 to N-1.

E — Find the freq-domain representation of x[n] given by



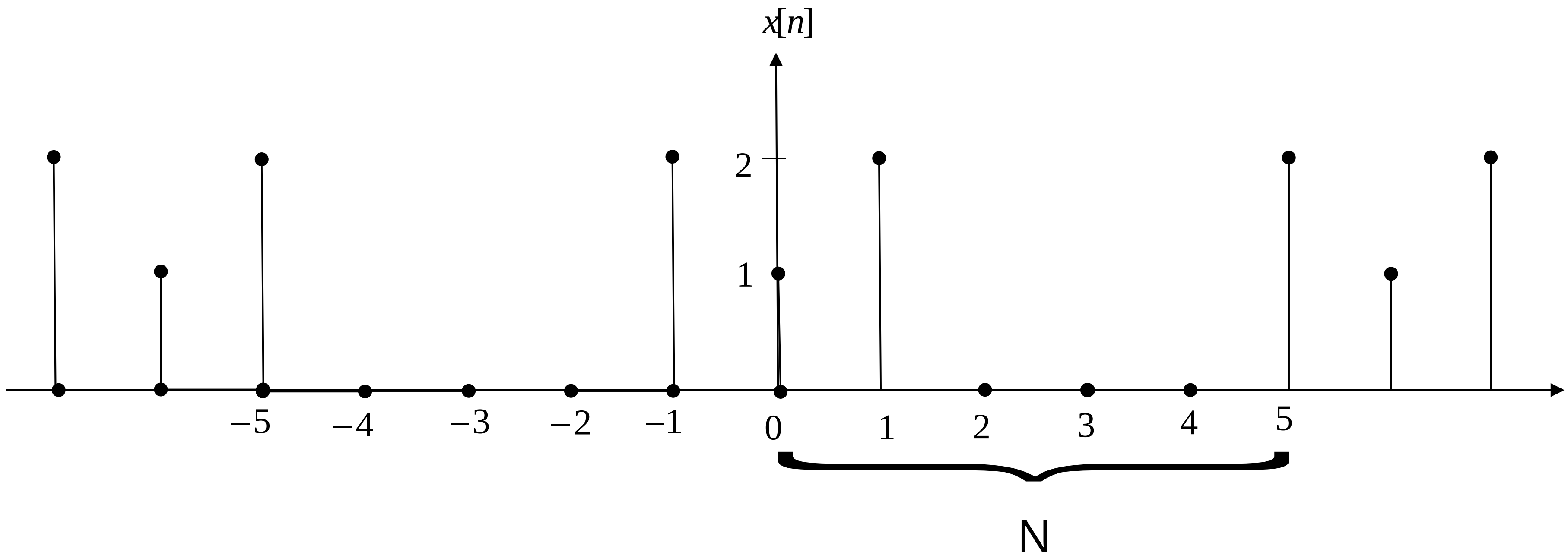
<u>Solution</u>:

$N = 5$ (Period)

$\Omega_0 = 2\pi/N = 2\pi/5$ (Fundamental frequency)

Solution:

E



Find DTFS coefficients X[k] of periodic signal x[n]

Solution:

E

$x[n] = \cos(\pi n \;\, 3\phi +)$, Find $X[k]$. /

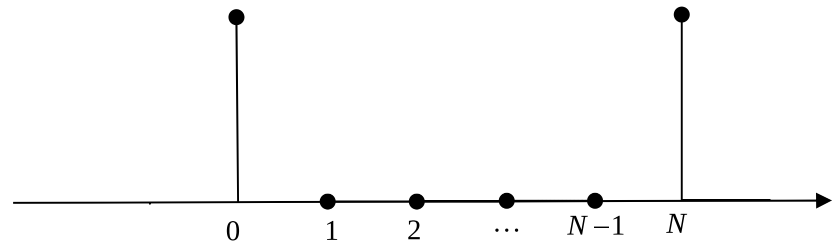
E

$$x[n] = 1 + \sin(n\pi/12 + 3\pi/8)$$

E

DTFS of an Impulse train: $x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN]$

Solution:

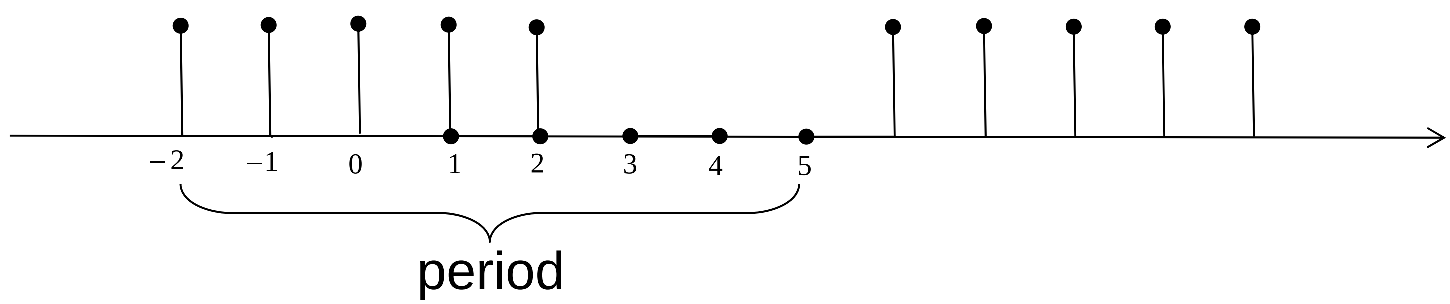


E

DTFS of a square signal:

$$x[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & M < n < N - M \end{cases}$$

$$\text{Example} \begin{cases} M = 2 \\ N = 8 \end{cases}$$



Solution:

E One period of DTFS coefficients

$$X[k] = \left(\frac{1}{2}\right)^k, \; 0 \le k \le 9$$

Determine $x[n]$ assuming $N = 10$

Solution:

<u>F.S.</u> (CT, periodic). x(t): fundamental period T

fundamental frequency

$$\omega_0 = 2\pi/T$$

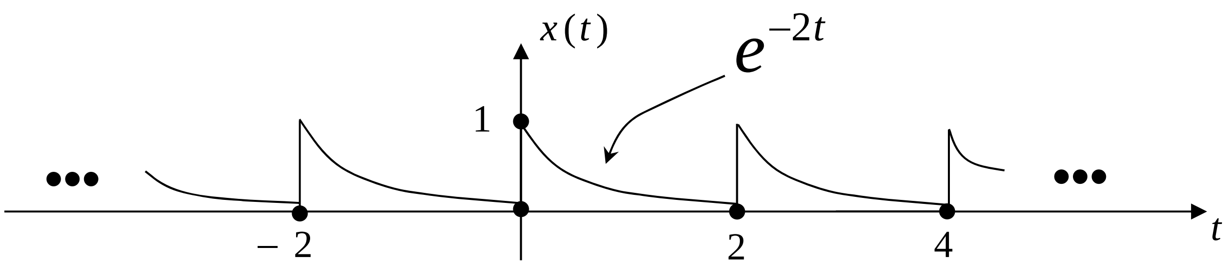
$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} & (*) \\ X[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \end{cases}$$

x(t) and X[k] are an FS pair: $\boxed{x(t) \xleftrightarrow{FS;\omega_0} X[k]}$

FS coefficients X[k] are a freq-domain representation of x(t).

E

x(t) given as



Solution:

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \qquad \text{where } \begin{cases} T = 2 \\ \omega_0 = \pi = \dfrac{2\pi}{T} \end{cases}$$

$$= \frac{1}{2}\int_0^2 e^{-2t}e^{-j\pi kt}dt$$

$$= \frac{1}{2}\int_0^2 e^{-(2+j\pi k)t}dt$$

$$= \frac{-1}{2(2+j\pi k)} e^{-(2+j\pi k)t}\Big|_0^2$$

$$= \frac{1}{4+j2\pi k}\left(1-e^{-4}e^{-j2\pi k}\right)$$

$$= \frac{1-e^{-4}}{4+j2\pi k} \qquad = 1$$

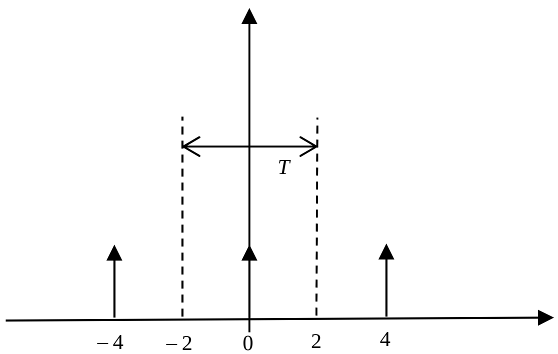
$$|X[k]|$$

$$X[k]$$

E

Determine X[k] of $x(t)=\sum_{l=-\infty}^{\infty}\delta(t-4l)$

Solution:



FS coefficients by inspection.

E

$$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$

Find X[k]

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$
$$= \sum_{k=-\infty}^{\infty} X[k] e^{jk(\pi/2)t}$$

Solution:

E

$x(t) = 2\sin(2\pi t - 3+)\sin(6\pi t)$. Find $X[k]$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

Solution:

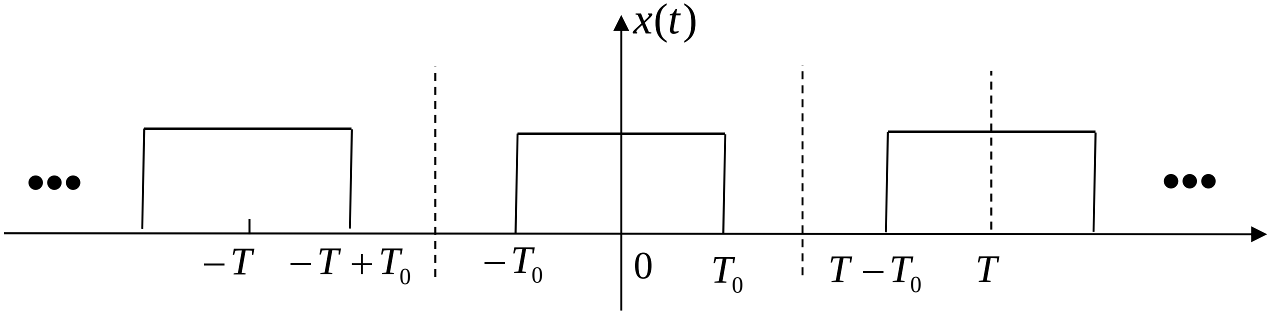
E Inverse FS.

$$X[k] = -j\delta[k-2] + j\delta[k+2] + 2\delta[k-3] + 2\delta[k+3], \; \omega_0 = \pi. \text{ Find } x(t)$$

Solution:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \\ &= -je^{j2\pi t} + je^{-j2\pi t} + 2e^{j3\pi t} + 2e^{-j3\pi t} \\ &= 2\sin(2\pi t) + 4\cos(3\pi t) \end{aligned}$$

E FS of a square wave.



Period is T, so $\omega_0 = 2\pi/T$

Solution:

DTFT D.T., nonperiodic

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

DTFT of signal x[n], also Freq-domain representation of x[n].

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$$

$$X(e^{j(\Omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \cdot e^{-j2\pi n} = X(e^{j\Omega})$$

E

$$x[n] = \alpha^n u[n], \quad X(e^{j\Omega}) = ?$$

Solution:

E

rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases} \qquad X(e^{j\Omega}) = ?$$

Solution:

E

Inverse DTFT of a rectangular spectrum, Example 3.19, p_{234}

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$

$X(e^{j\Omega})$ is defined over $(-\pi, \pi)$
periodic in Ω

Solution:

E DTFT of unit impulse $x[n] = \delta[n]$

Solution:

- What about inverse DTFT of a unit impulse spectrum?

$$X(e^{j\Omega}) = \delta[\Omega], \quad -\pi < \Omega \leq \pi \qquad \text{(defined only one period)}$$

Solution:

E

$$x[n] = \begin{cases} 2^n, & 0 \le n \le 9 \\ 0, & o.w. \end{cases} \qquad X(e^{j\Omega}) = ?$$

Solution:

E

$$X(e^{j\Omega}) = 2\cos(2\Omega), \quad x[n] = ?$$

Use inspection! $X(e^{j\Omega}) = e^{j2\pi} + e^{-j2\pi}$

FT C.T., nonperiodic signals

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\omega\right)e^{j\omega t}dt \\ X\left(j\omega\right) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \end{cases}$$

$$x(t) \stackrel{FT}{\longleftrightarrow} X(j\omega)$$

E

$$x(t) = e^{-at}u(t). \quad \text{Find} \quad X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

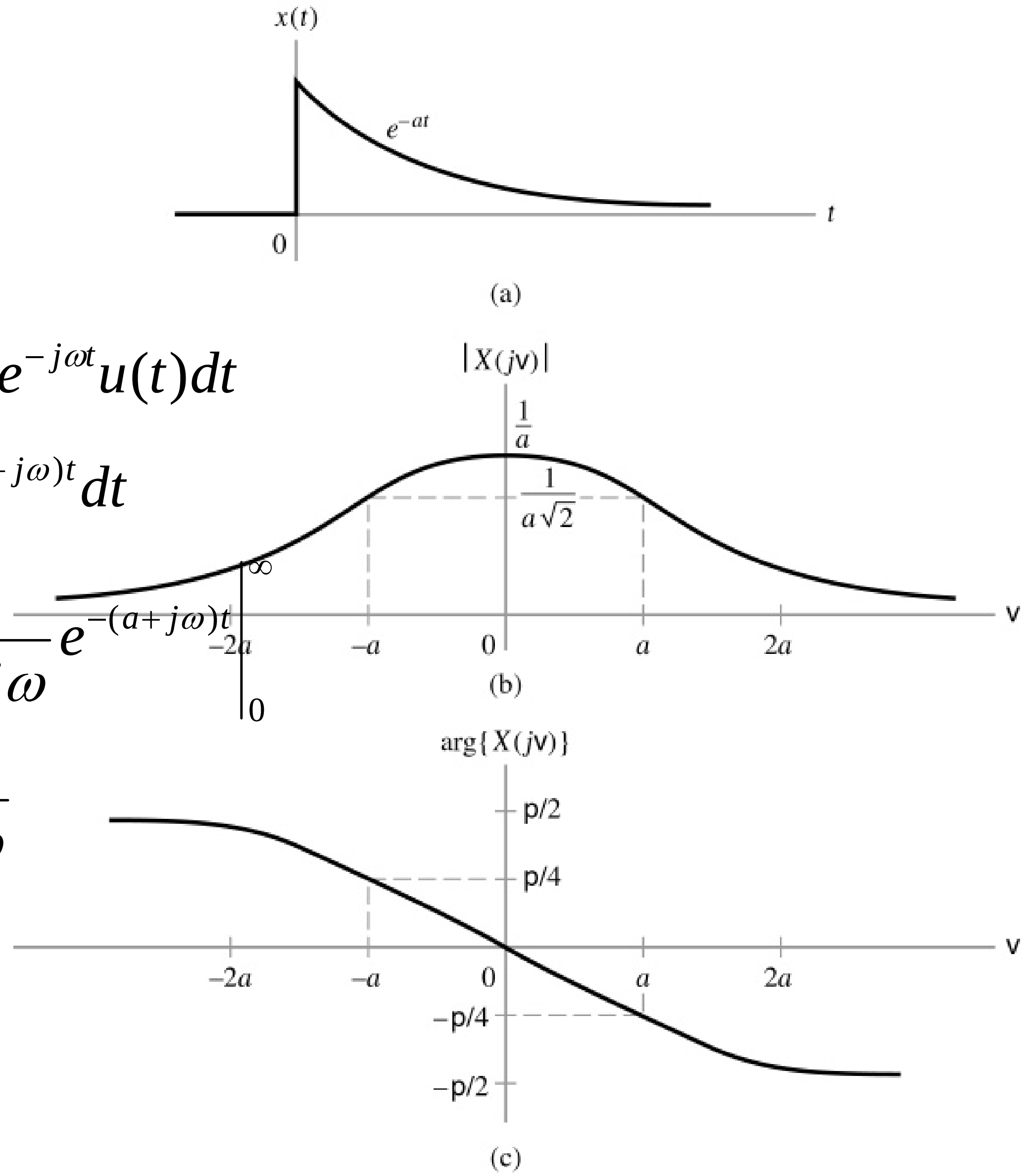
Solution:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} e^{-j\omega t} u(t)\,dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}$$

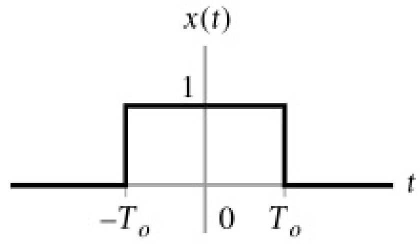


E

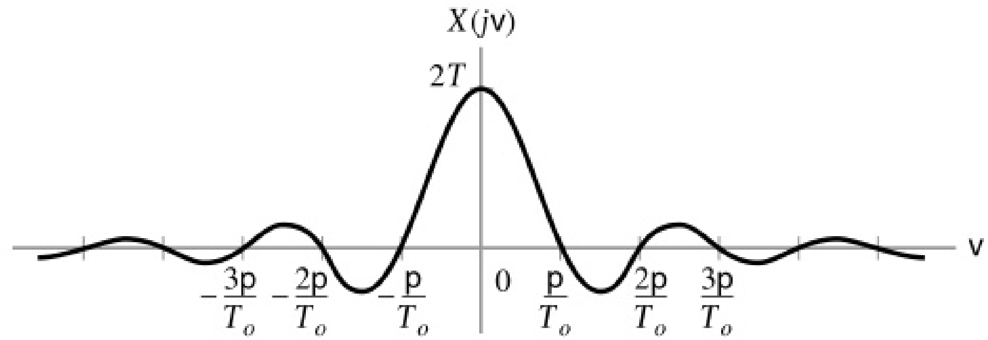
Rectangular pulse:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases}$$

Solution:



(a)



(b)

E

Inverse FT of a rectangular spectrum:

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

E

Unit impulse: $x(t) = \delta(t)$

E

Inverse FT of an impulse spectrum: $X(j\omega) = 2\pi\delta(\omega)$

PROPERTIES OF FOURIER REPRESENTATIONS

Time property	Periodic (t,n)	Nonperiodic (t,n)	
C.T. (t)	• Fourier Series (FS) $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t},\ \omega_0=\frac{2\pi}{T}$ $X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt,$ (T: period)	• Four Transform (FT) $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$	Non-periodic (k,ω)
D.T. [n]	• Discrete-Time Fourier Series (DTFS) $x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n},\ \Omega_0=\frac{2\pi}{N}$ $X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_0 n}$ (N: period)	• Discrete-Time Fourier Transform (DTFT) $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$	Periodic (k,Ω)
	Discrete [k]	Continuous (ω, Ω)	Freq. property

- Linearity and symmetry

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_0} Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

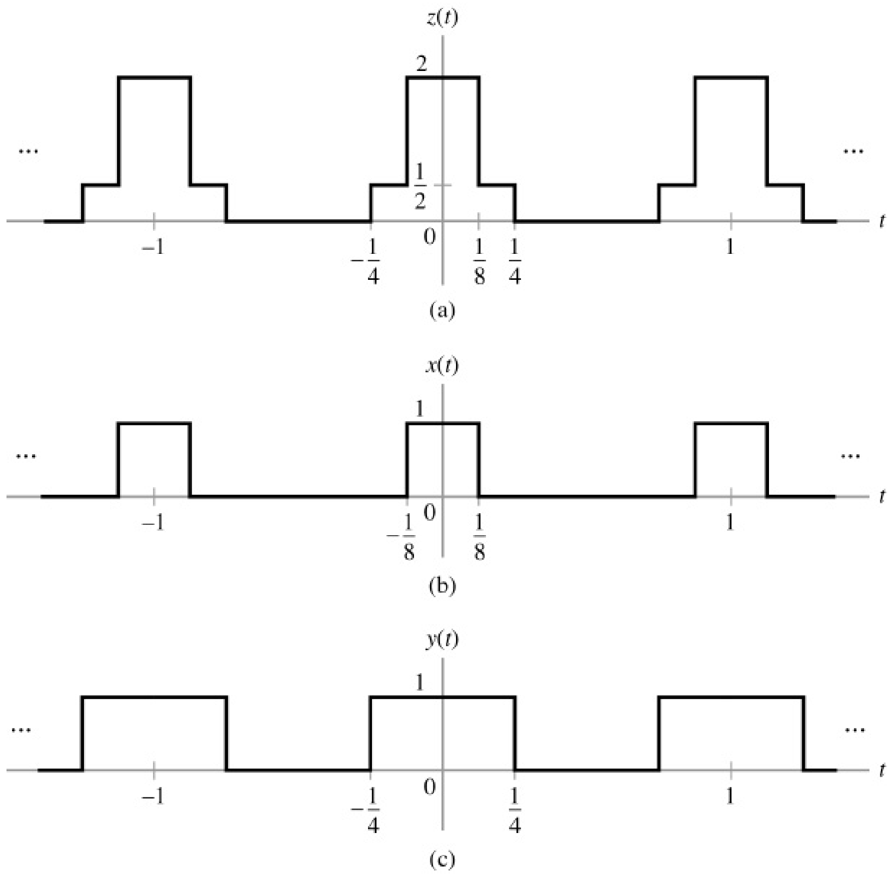
$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_0} Z[k] = aX[k] + bY[k]$$

E

Example 3.30, p255:

$$z(t) = \frac{3}{2}x(t) + \frac{1}{2}y(t)$$

Find the frequency-domain representation of z(t).



Which type of freq.-domain representation?

- FT, FS, DTFT, DTFS ?

Solution:

Symmetry:

We will develop using continuous, non-periodic signals. Results for other 2 cases may be obtained in a similar way.

a) Assume $x(t)\ \text{real} \Rightarrow x^*(t) = x(t)$

$$\text{If } x(t) \text{ is real} \Rightarrow X(j\omega) \text{ is conjugate symmetric}$$

Proof:

If $x(t)$ is real and even $\Rightarrow X(j\omega)$ is real.

Proof:

If $x(t)$ is real and odd $\Rightarrow X(j\omega)$ is purely imaginary.

Proof:

If $x(t)$ is purely imaginary $\Rightarrow$

- Real part of $X(j\omega)$ has odd symmetry
- Imaginary part of $X(j\omega)$ has even symmetry

- <u>Convolution:</u> Applied to non-periodic signals.

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

Proof:

E

Let $x(t) = \frac{1}{\pi t}\sin(\pi t)$ be input to a system with impulse response $h(t) = \frac{1}{\pi t}\sin(2\pi t)$. Find the system output $y(t)$

Solution:

E

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega). \qquad \text{Find } x(t).$$

Solution:

The same convolution properties hold for discrete-time, non-periodic signals.

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

- Differentiation and integration:
  - Applicable to continuous functions: time (t) or frequency (ω or Ω)
  - FT (t, ω) and DFTF (Ω)

Differentiation in time:

$$\frac{d}{dt}x(t) \stackrel{FT}{\longleftrightarrow} j\omega X(j\omega)$$

Proof:

E Find FT of $\frac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0$

Solution:

E Find x(t) if $X(j\omega) = \begin{cases} j\omega, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$

Solution:

If x(t) is periodic, frequency-domain representation is Fourier Series (FS):

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$\frac{d}{dt} x(t) \xleftrightarrow{FS;\omega_0} jk\omega_0 X[k]$$

Differentiation in frequency:

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$$

Proof:

E

A Gaussian pulse is given as: $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$. Find its FT.

Solution:

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$$

E Determine the Fourier transform of u(t).

Solution:

E   Find x(t), given

$$X(j\omega) = \frac{1}{j\omega(j\omega + 1)} + \pi\delta(j\omega)$$

Solution:

E

$$x(t) = \frac{d}{dt}\left(2te^{-2t}u(t)\right). \qquad X(j\omega) = ?$$

$$-jtx(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega}X(j\omega)$$

$$\frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega X(j\omega)$$

- $e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{2 + j\omega}$
- $te^{-2t}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{(2 + j\omega)^2}$
- $\dfrac{d}{dt}\left(2te^{-2t}u(t)\right) \overset{FT}{\longleftrightarrow} \dfrac{2j\omega}{(2 + j\omega)^2}$

- Time and frequency shift

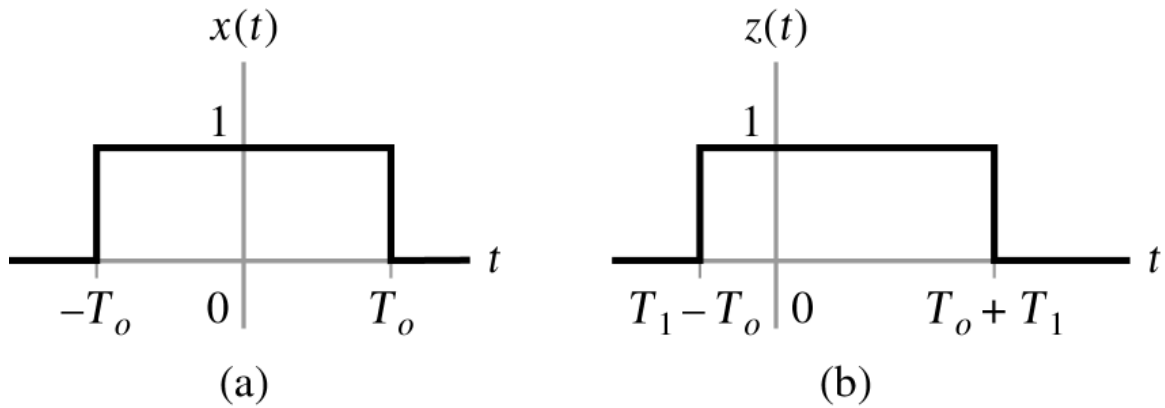
Time shift: $$Z(j\omega) = e^{-j\omega t_0} X(j\omega)$$

Proof:

Note: Time shift $\Rightarrow$ phase shift in frequency domain. Phase shift is a linear function of ω. Magnitude spectrum does not change.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$
$$x(t-t_0) \xleftrightarrow{FS;\omega_0} e^{-jk\omega t_0} X[k]$$
$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$
$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega n_0} X[k]$$

E Find Z(jω)



$$z(t) = x(t-T_1)$$
$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$
$$z(t) \xleftrightarrow{FT} Z(j\omega) = e^{-j\omega T_1}\frac{2}{\omega}\sin(\omega T_0)$$

E $$X(j\omega) = \frac{e^{j4\omega}}{(2 + j\omega)^2}.$$ Find $x(t)$

Solution:

Frequency shift:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$X(j(\omega-\gamma)) \xleftrightarrow{FT} e^{j\gamma t}x(t)$$

Proof:

Note:
– Frequency shift $\Rightarrow$ time signal multiplied by a complex sinusoid.
– Carrier modulation.

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$
$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\omega_0} X[k-k_0]$$
$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$
$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{DTFS;\Omega_0} X[k-k_0]$$

E

$$z(t)=\begin{cases} e^{j10t}, & |t|<\pi \\ 0, & |t|>\pi \end{cases}$$

Find Z(jω).

Solution:

E $x(t) = \frac{d}{dt}\left\{\left(e^{-3t}u(t)\right) * \left(e^{-t}u(t-2)\right)\right\}.$ Find $X(j\omega)$.

Solution:

• Multiplication

CT, non-periodic

$$y(t) = x(t)z(t) \stackrel{FT}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi} X(j\omega) * Z(j\omega)$$

DT, non-periodic

$$y[n] = x[n]z[n] \stackrel{DTFT}{\longleftrightarrow} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

Periodic convolution: $X(e^{j\Omega}) \circledast Z(e^{j\Omega}) = \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Z\left(e^{j(\Omega-\theta)}\right) d\theta$

CT, periodic

$$y(t) = x(t)z(t) \stackrel{FS;2\pi/T}{\longleftrightarrow} Y[k] = X[k] * Z[k]$$

DT, periodic

$$y[n] = x[n]z[n] \stackrel{DTFS;2\pi/N}{\longleftrightarrow} Y[k] = X[k] \circledast Z[k]$$

$$X[k] \circledast Z[k] = \sum_{m=0}^{N-1} X[m]Z[k-m]$$

- Scaling $\quad x(at) \leftrightarrow \frac{1}{|a|} X(j(\omega / a))$

Proof:

E

$$x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

$$y(t) = \begin{cases} 1 & |t| < 2 \\ 0 & |t| > 2 \end{cases}$$

Find $Y(j\omega)$

Solution:

E

Find $x(t)$ if $X(j\omega) = j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

We know $s(t) = e^{-t}u(t) \overset{FT}{\longleftrightarrow} S(j\omega) = \dfrac{1}{1+j\omega}$

– Time scaling: $z(t) = s(3t) \overset{FT}{\longleftrightarrow} Z(j\omega) = \dfrac{1}{3}\dfrac{1}{1+j(\omega/3)}$

– Time shift: $v(t) = 3z(t+2) = 3s(3(t+2)) \overset{FT}{\longleftrightarrow} \dfrac{e^{j2\omega}}{1+j(\omega/3)}$

–Differentiation: $tv(t) \overset{FT}{\longleftrightarrow} j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

Thus,

$$
\begin{aligned}
x(t) &= tv(t) \\
&= 3tz(t+2) \\
&= 3ts(3(t+2)) \\
&= 3te^{-3(t+2)}u(3(t+2))
\end{aligned}
$$

$$x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|}X(j(\omega/a))$$

$$x(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$$

$$-jtx(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega}X(j\omega)$$

Note: $u(3(t+2)) = u(t+2)$. Thus, $x(t) = 3te^{-3(t+2)}u(3(t+2))$

- <u>Parseval's relationship:</u>

Energy of CT, non-periodic signal $x(t)$: $W_x = \int_{-\infty}^{\infty} | x(t) |^2 dt$

$$= \int_{-\infty}^{\infty} x^*(t) x(t) dt$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega$$

$$W_x = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) d\omega$$

$$\boxed{W_x = \int_{-\infty}^{\infty} | x(t) |^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} | X(j\omega) |^2 \, d\omega}$$

Note: a) $| X(j\omega) |^2$: energy spectrum

b) Energy in time domain = energy in freq. domain

- DTFT: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$
- DTFS: $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$
- FS: $\frac{1}{T}\int_{0}^{T} |x(x)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$

E

$$x[n] = \frac{\sin(Wn)}{\pi n}. \qquad \text{Determine } E_x = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{(\pi n)^2}$$

$$x[n] = \frac{\sin(Wn)}{\pi n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| \le \pi \end{cases}$$

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = W/\pi$$

• Time-bandwidth product

Compression in time domain ⇒ expansion in frequency domain

Bandwidth: The extent of the signal's significant contents. It is in general a vague definition as "significant" is not mathematically defined. In practice, definitions of bandwidth include

– absolute bandwidth
– x% bandwidth
– first-null bandwidth.

If we define

$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right]^{1/2} \quad : \quad \text{RMS duration of an energy signal}$$

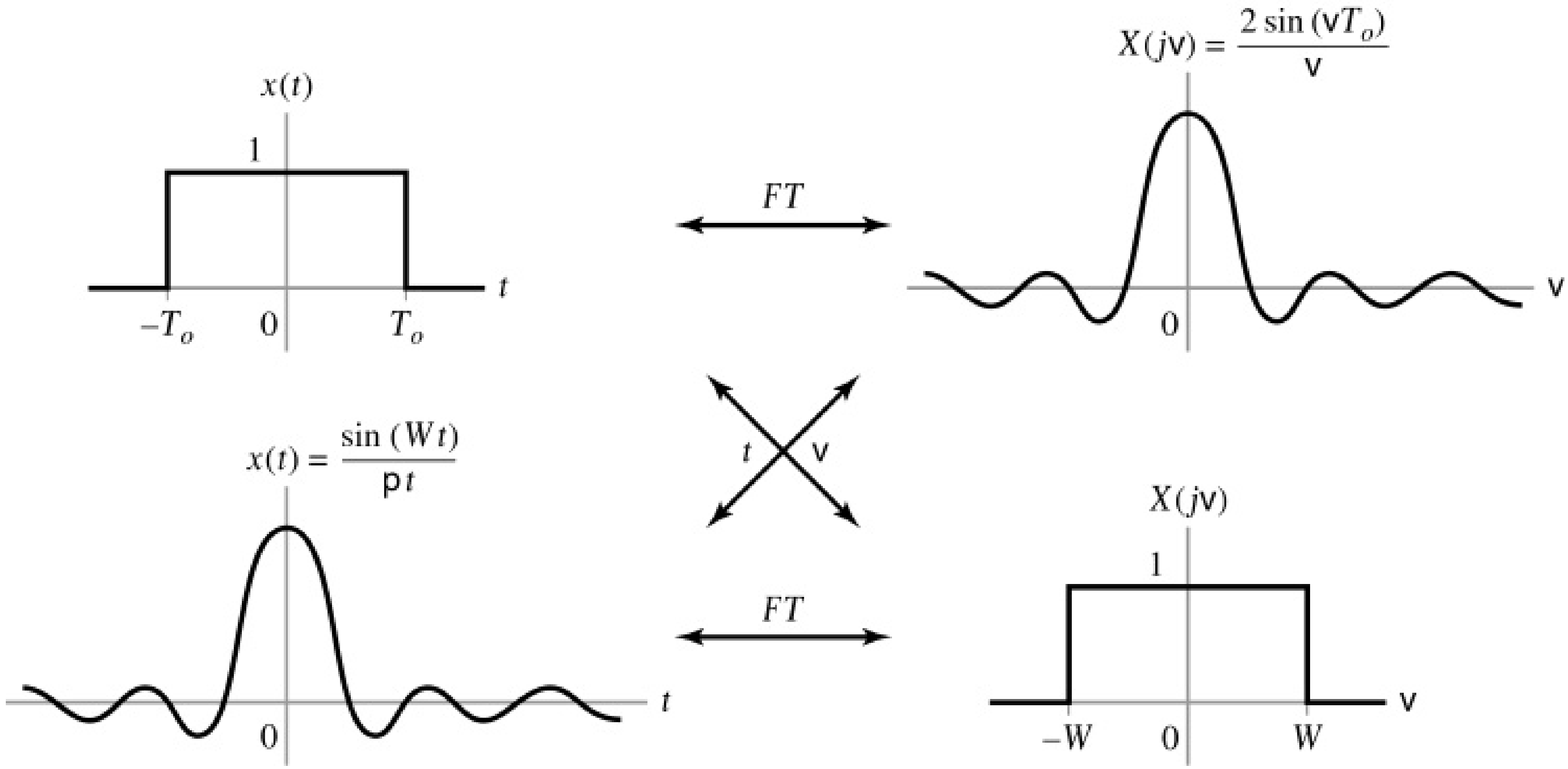
$$B_w = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{1/2} \quad : \quad \text{RMS bandwidth, then}$$

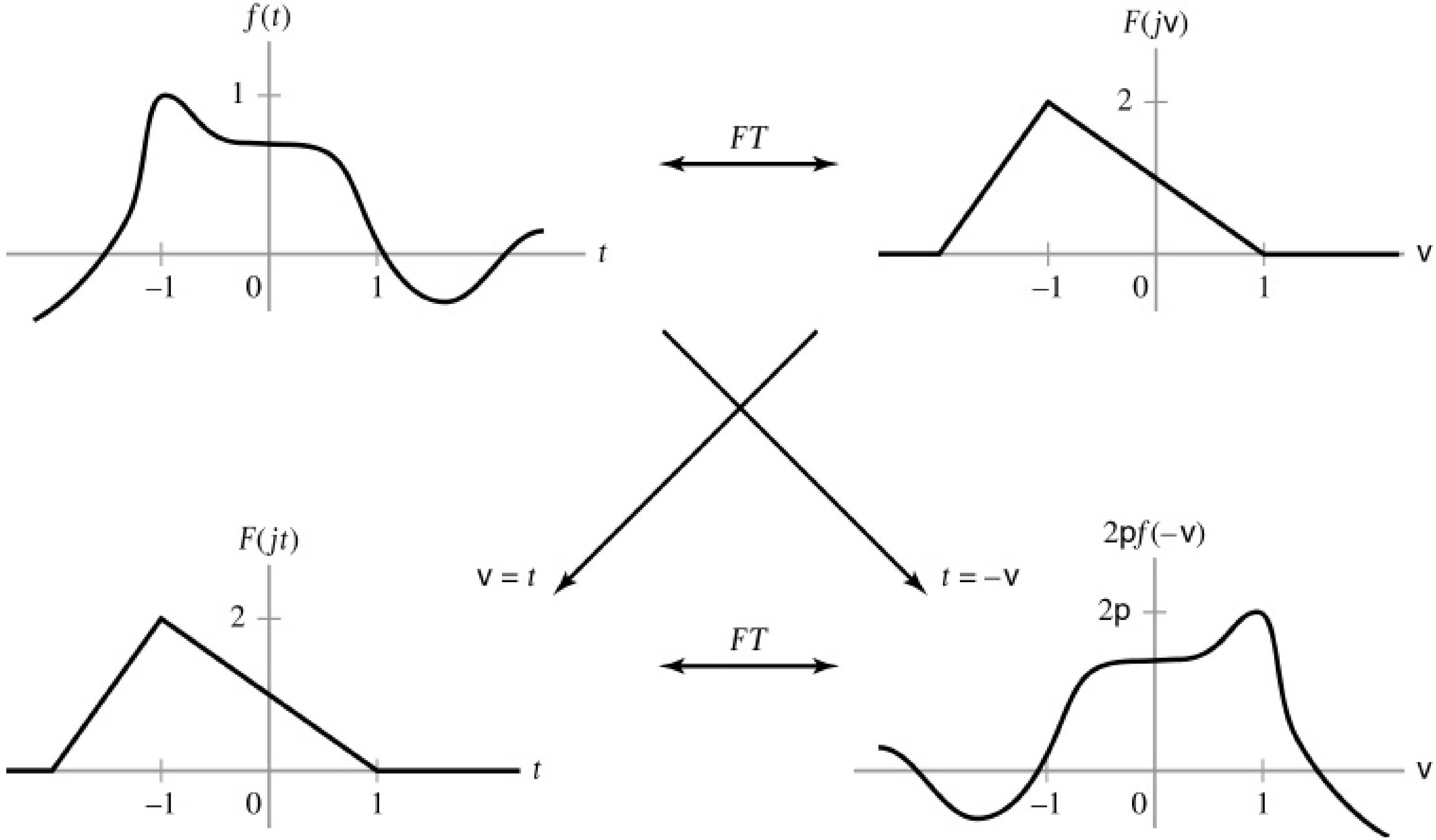
$$T_d B_w \geq 1/2$$

- Duality

$$f(t) \overset{FT}{\longleftrightarrow} F(j\omega)$$

$$F(t) \overset{FT}{\longleftrightarrow} 2\pi f(-\omega)$$





f(t)
1
t
−1
0
1
FT
F(jv)
2
v
−1
0
1
F(jt)
2
v = t
t = −v
2pf(−v)
2p
FT
t
−1
0
1
v
−1
0
1

E

Find $X(j\omega)$ if $x(t) = \dfrac{1}{1+jt}$

We know $f(t) = e^{-t}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{1+j\omega} = F(j\omega)$

Duality $F(jt) = \dfrac{1}{1+jt} \overset{FT}{\longleftrightarrow} 2\pi f(-\omega) \Rightarrow$

$$X(j\omega) = 2\pi f(-\omega) = 2\pi e^{\omega}u(-\omega)$$

Check $x(\mathrm{t}) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} 2\pi e^{\omega}e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{0} e^{\omega(1+jt)}d\omega = \frac{1}{1+jt}$$

E

Find $x(t)$ if $X(j=)\omega u(\omega)$

$$X(j\omega) = u(\omega)$$

$$X(jt) = u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(-\omega) = 2\pi x(-\omega) \Rightarrow$$

$$x(t) = \left[\frac{1}{j(-t)} + \pi\delta(-(-t))\right]\frac{1}{2\pi}$$

$$= \frac{-1}{2\pi jt} + \frac{1}{2}\delta(t)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

- DTFS: $\begin{cases} x[n] \xleftrightarrow{DTFS;2\pi/N} X[k] \\ X[n] \xleftrightarrow{DTFS;2\pi/N} \frac{1}{N}x[-k] \end{cases}$
- DTFT and FS: $\begin{cases} x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) \\ X\left(e^{jt}\right) \xleftrightarrow{FS;1} x[-k] \end{cases}$
- *D*TFT and FS do not stay in their own class!