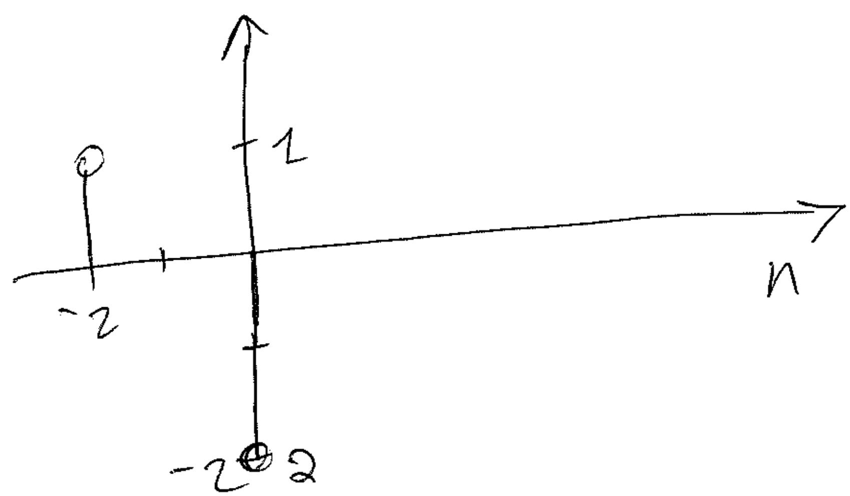
b) If an LTI system is CAUSAL: $\Longleftrightarrow$ $\begin{array}{l} \text{DT}: h[k]=0 \text{ for } k<0 \\ \text{CT}: h(\tau)=0 \text{ for } \tau<0 \end{array}$

Proof:

ex:

$$y[n] = x[n+2] - 2x[n]$$

$$h[n] = \delta[n+2] - 2\delta[n]$$



$\Rightarrow$ non-causal.

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c) If an LTI system is BIBO STABLE: $\longleftrightarrow$ $\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Proof:

BIBO stable: $|x[n]| < M_x \xrightarrow{H} |y[n]| < M_y$

$M_x, M_y < \infty$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| < \left|\sum_{k=-\infty}^{\infty} h[k]\right| M_x \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\le M_y \Rightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$(|a+b| \le |a| + |b|)$

E First-order autoregressive system

$$y[n] = \rho y[n-1] + x[n], \text{ with } h[n] = 0 \text{ for } n < 0$$

Substitute $x[n] = \delta[n]$

$h[n] = \rho\, h[n-1] + \delta[n]$

$h[0] = \rho\, h[-1] + \delta[0] = (\rho)(0) + 1 = 1$

$h[1] = \rho\, h[0] + \delta[1] = \rho$

$h[2] = \rho\, h[1] + \delta[2] = \rho^2$

$h[n] = \rho^n u[n]$

- Memory / memoryless? = Memory. $(\rho^n u[n] \neq c\,\delta[n])$
- Causal / Non causal = Causal $(h[n] = 0,\ n < 0)$
- $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\rho^n u[n]| = \sum_{n=0}^{\infty} |\rho^n| = \sum_{n=0}^{\infty} |\rho|^n$

$|\rho| < 1 \Rightarrow \frac{1}{1-|\rho|}$ BIBO stable ✓, $|\rho| \geq 1 \Rightarrow$ not BIBO stable.

STEP RESPONSE:

LTI

$$\begin{matrix} x[n] \\ x(t) \end{matrix} \rightarrow \boxed{H} \rightarrow \begin{matrix} y[n] \\ y(t) \end{matrix}$$

Step response: if $x[n] = u[n] \Rightarrow y[n] = s[n] = \sum_{k=-\infty}^{n} h[k]$

Proof:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

let $x[n-k] = u[n-k]$

$$= \sum_{k=-\infty}^{\infty} h[k]\, u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

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Example

$$u[n] \rightarrow \boxed{h[n] = \rho^n u[n]} \rightarrow y[n] = ?? \qquad |\rho| < 1$$

$$y[n] = s[n]$$

$$s[n] = \sum_{k=-\infty}^{n} h[k] = \sum_{k=\cancel{0}\infty}^{n} \rho^k u[k]$$

$$= \sum_{k=0}^{n} \rho^k = \frac{1-\rho^{n+1}}{1-\rho}$$