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DIFFERENTIAL & DIFFERENCE EQUATION REPRESENTATIONS OF LTI SYSTEMS

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

$$x[n] \qquad\qquad y[n]$$

Input-output relation can be described as

CT $$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Linear constant-coefficient differential equation.

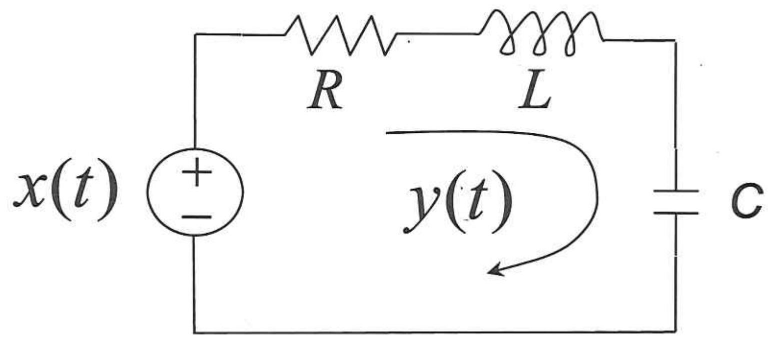
DT $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Linear constant-coefficient difference equation.

a_k, b_k constants

- Order of differential/difference equation: *(N,M)*
- Often N ≥ M, and <u>order is described using *N* only</u>

E Describe the following RLC circuit by a differential equation.



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{c}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$ differentiates both sides

$$\frac{1}{c}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t)$$

$N=2$ $\quad (a_0, a_1, a_2) = (\frac{1}{c}, R, L)$

$M=1$ $\quad (b_0, b_1) = (0, 1)$

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E

2nd-order difference equation

$$y[n] + y[n-1] + \frac{1}{4} y[n-2] = x[n] + 2x[n-1]$$

$$\begin{cases} (a_0, a_1, a_2) = (1,\ 1,\ 1/4) \\ (b_0, b_1) = (1\ ,2) \end{cases}$$

y[n] can be evaluated recursively:

Need *y[-2], y[-1]:* Initial conditions. *x[-1]* depends on input applied.

For example, if $x[n] = \left(\tfrac{1}{2}\right)^n u[n], y[-1] = 1, y[-2] = -2$, then

$$y[0] = 1 + 2 \times 0 - 1 - \frac{1}{4} \times (-2) = \frac{1}{2}$$

$$y[1] = \frac{1}{2} + 2 \times 1 - \frac{1}{2} - \frac{1}{4} \times (1) = 1\frac{3}{4}$$

$\vdots$

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4} y[n-2]$$

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4} y[-2]$$

$$= 1 + 0 - 1 + \frac{2}{4} = \frac{1}{2}$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4} y[-1]$$

$$= \frac{1}{2} + 2 - \frac{1}{2} - \frac{1}{4} = 1\frac{3}{4}$$

$$y[2] =$$

SOLVING DIFFERENTIAL AND DIFFEENCE EQUATIONS

Just a review, rather than in depth:

Complete Solution

Solution forms:	Homogeneous		Particular
CT	$y(t) = y^{(h)}(t)$	$+$	$y^{(p)}(t)$
DT	$y[n] = y^{(h)}[n]$	$+$	$y^{(p)}[n]$
	$\Uparrow$		$\Uparrow$
	Natural response		Forced response
	$\boxed{\text{Input signal } x(t) \text{ or } x[n] = 0}$		$\boxed{\text{Initial rest}}$
	Depends on initial condition!		Depends on input signal!

GENERAL CASE

CT system: $y^{(h)}(t)$ is the solution of the *homogeneous* equation:

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0}$$

The homogeneous *solution* is of the form: $y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t}$

where r_i are the N roots of the system's *characteristic* equation

$$\boxed{\sum_{k=0}^{N} a_k r^k = 0}$$

Note: c_i: to be determined (later) so that the complete solutions satisfy the initial conditions.

Example

$x[n] = 0$

- Homogeneous equation: $y[n] - \frac{1}{4} y[n-1] = 0$ (set *x[n]*=0)

$N = 1$, $(a_0, a_1) = (1, -\frac{1}{4})$

- Characteristic equation: $\sum_{k=0}^{N} a_k r^{N-k} = 0$

$(1) r^{1-0} + (-\frac{1}{4}) r^{1-1} = 0 \Rightarrow r = \frac{1}{4}$

- Solution of homogeneous equation: $y^{(h)}[n] = c_1 (\frac{1}{4})^n$

c_1: to be determined so that the complete solutions satisfy the initial conditions.

- A particular solution is assumed independent of the homogeneous solution
- Usually obtained assuming that output has the same form as input signals.
- The form of the particular solution associated with common inputs are summarized in the following table.

Continuous time		Discrete time	
Input	Particular solution	Input	Particular solution
1	c	1	c
t	c_1t+c_2	n	c_1n+c_2
e^{-at}	$c\,e^{-at}$	α^n	$c\,\alpha^n$
$\cos(\omega t+\phi)$	$c_1\cos(\omega t+\phi)+c_2\sin(\omega t+\phi)$	$\cos(\Omega n+\phi)$	$c_1\cos(\Omega n+\phi)+c_2\sin(\Omega n+\phi)$

INPUT~ PARTICULAR FORM