FOURIER REPRESENTATION OF SIGNALS & LTI SYSTEMS

CT: f cycle/second (Hz)
$\omega = 2\pi f$ rads/s

DT: F cycles/sample
$\Omega = 2\pi F$ rads/sample

- Basic signals as weighted superposition of impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xrightarrow[h[n]]{LTI} y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

superposition weight delay (LTI property)

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow[h(t)]{LTI} y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Time-domain waveform represents how fast signal changes. Signals in terms different frequency components or weighted superpositions of complex sinusoids.

$$\begin{cases} \text{CT: } X(f) \text{ or } X(\omega) \\ \text{DT: } X[k] \end{cases}$$

Why signals represented as weighted superpositions of complex sinusoids?

DT: $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$x[n] = e^{j\Omega n}$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega(n-k)} = \sum_{k=-\infty}^{\infty} h[k] \left(e^{j\Omega n}\right)\left(e^{-j\Omega k}\right)$$

$$= e^{j\Omega n}\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}\right)$$

$$= e^{j\Omega n} H\left(e^{j\Omega}\right)$$

(Note: $H(e^{j\Omega})$ can also be written as $H(j\Omega)$, $H(\Omega)$)

$$\boxed{H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}}$$ ~ related to h[k]

Note:

a). $H(e^{j\Omega})$ is NOT a function of *n*, only a function of Ω.

$H(e^{j\Omega})$ Is called the frequency response.

b). System modifies the amplitude of input by $|H(e^{j\Omega})|$.

$|H(e^{j\Omega})|$: magnitude response.

c). System introduces a phase lag $\angle H(e^{j\Omega})$.

(the book uses $\arg\{H(e^{j\Omega})\}$)

$$\boxed{H(e^{j\Omega}) = |H(e^{j\Omega})| e^{j\angle H(e^{j\Omega})}}$$

$A = a + bj$

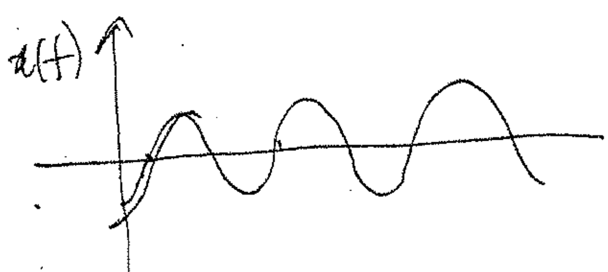
$= |A| e^{j\angle A}$

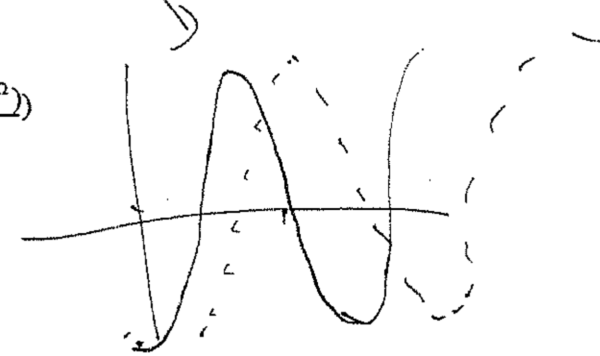
CT: $$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}$$

with $x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega) e^{j\omega t}$

$= |H(j\omega)| e^{j(\omega t + \angle H(e^{j\Omega}))}$

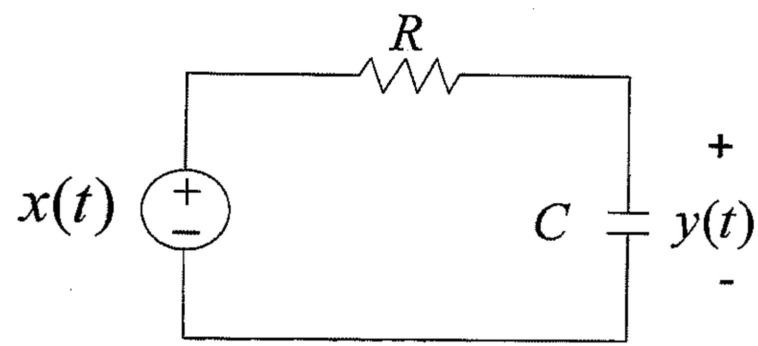
$y(t)$





same frequency

E Example



Impulse response: $\boxed{h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)}$

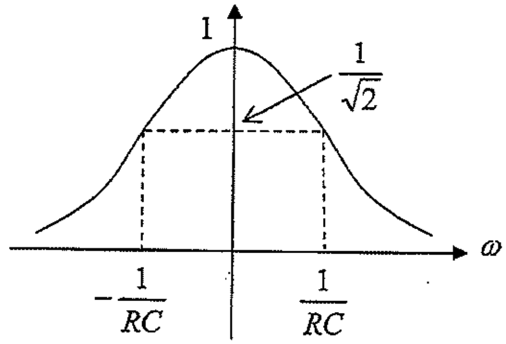
Find frequency response.

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\frac{\tau}{RC}} e^{-j\omega\tau} v(\tau) d\tau$$

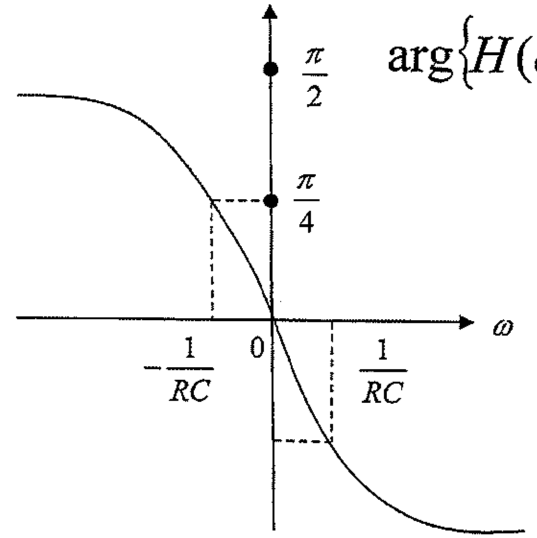
$$= \int_{0}^{\infty} \frac{1}{RC} e^{\left(-\frac{1}{RC} - j\omega\right)\tau} d\tau = \frac{1}{RC} \left(-\frac{1}{\left(j\omega + \frac{1}{RC}\right)} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} \Big|_{0}^{\infty} \right)$$

$$= \frac{1}{RC} \left(\frac{1}{j\omega + \frac{1}{RC}} \right)$$

$|H(j\omega)|$



$\arg\{H(e^{j\Omega})\} = \angle H(j\omega)$



$e^{j\omega t}$: eigenfunction of the LTI system (eigen value $\lambda = H(j\omega)$)

$(H\{e^{j\omega t}\} = \lambda e^{j\omega t})$

Now, if the input to an LTI system is expressed as a weighted sum of <u>*M* complex sinusoids</u>:

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}, \quad \text{then}$$

$$y(t) = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$$