Convolution sum

$$\begin{aligned} x(t) &= \delta(t) \xrightarrow{\mathcal{H}} y(t) = h(t) \\ x[n] &= \delta[n] \xrightarrow{\mathcal{H}} y[n] = h[n] \end{aligned}$$

where $h(t)$ (CT) and $h[n]$ (DT) are the system impulse responses.

$h(t)$ or $h[n]$ completely characterizes an LTI system.
By knowing $h(t)$ or $h[n]$, system output can be obtained for an arbitrary input signal $x(t)$ or $x[n]$.
How is $y(t)/y[n]$ related to $x(t)/x[n]$ and $h(t)/h[n]$?

Convolution sum (cont.)

We will start with DT systems, and then analyze CT systems.

- Any signal $x[n]$ can be expressed as a sum of time-shifted impulses as (shown graphically next slide)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$\delta[n] \xrightarrow{H} h[n]$$

$$x[k]\,\delta[n] \xrightarrow{H} x[k]\,h[n] \quad \text{(linear property)}$$

$$x[k]\,\delta[n-k] \xrightarrow{H} x[k]\,h[n-k] \quad \text{(Time invariant)}$$

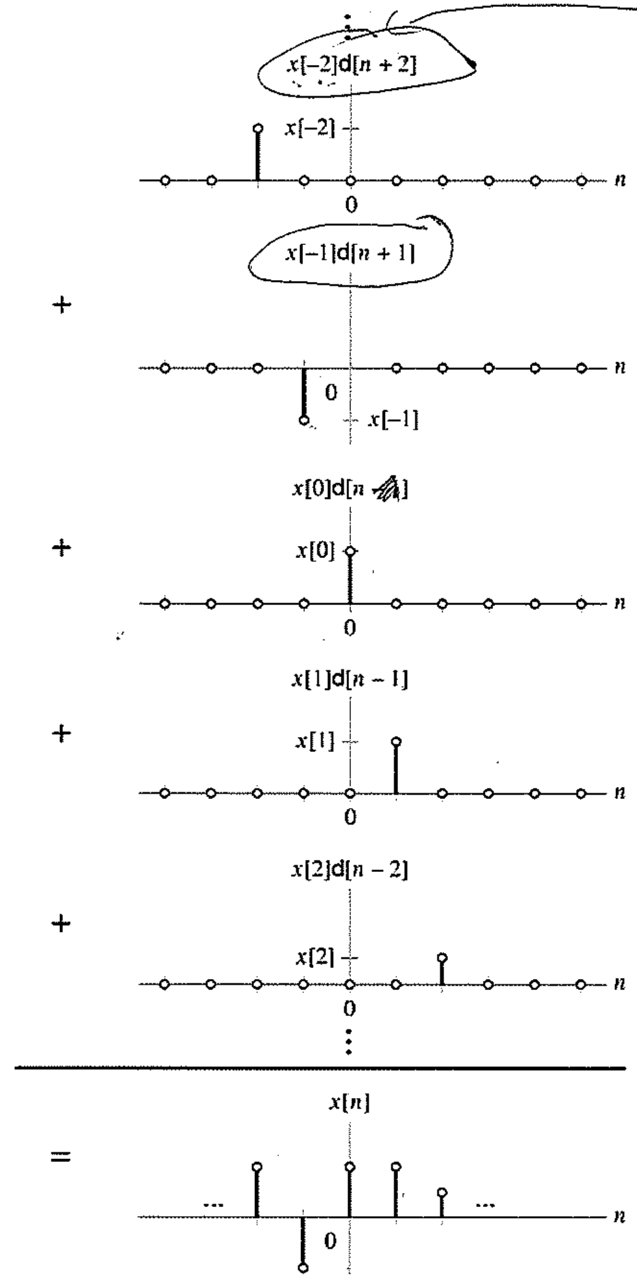
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \xrightarrow{H} \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] =: y[n] \quad \text{(linear property)}$$

Denote:

$$\underbrace{x[n] * h[n]}_{\text{convolution}} = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Convolution sum (cont.)

$d[n+2] = \delta[n+2]$



Convolution sum (cont.)

- Convolution sum: $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

All the usual properties such as associativity, commutativity and distributivity hold

- Properties of convolution

$x[n] * h[n] = h[n] * x[n]$ → show this yourselves.

$\delta[n] * h[n] = h[n]$ $= \sum_{k=-\infty}^{\infty} \delta[k] h[n-k] = h[n]$

$\delta[n-k] * h[n] = h[n-k]$ prove this yourselves.

E: A system with input-output relationship as

$$y[n] = x[n] + (1/2)x[n-1]$$

a) System impulse response?

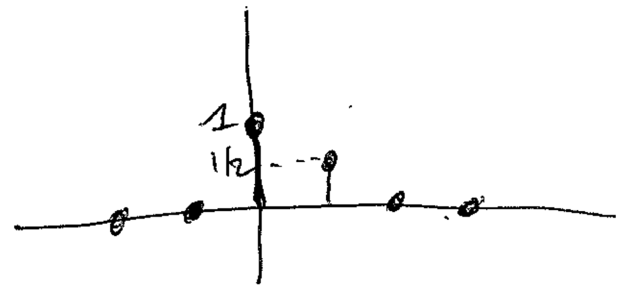
b) Find $y[n]$ for

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & o.w. \end{cases}$$

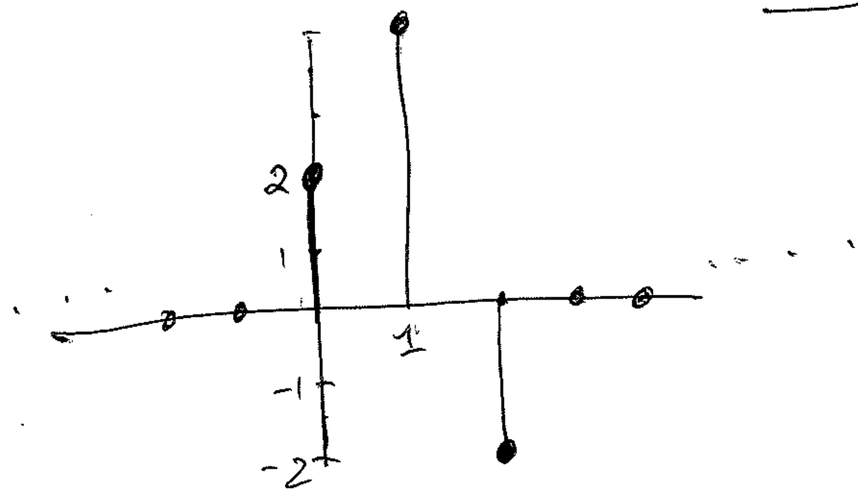
Convolution sum (cont.)

a) • replacing $x[n]$ with $\delta[n]$

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$



b) $x[n]$



$$x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$

$$y[n] = x[n] * h[n] = \left(2\delta[n] + 4\delta[n-1] - 2\delta[n-2]\right) * \left(\delta[n] + \frac{1}{2}\delta[n-1]\right)$$

$$2\delta[n] * \delta[n] + 2\delta[n] * \frac{1}{2}\delta[n-1] + 4\delta[n-1] * \delta[n] + 2\delta[n-1] * \delta[n-1]$$

$$- 2\delta[n-2] * \delta[n] - \delta[n-2] * \delta[n-1]$$

$$= 2\delta[n] + \delta[n-1] + 4\delta[n-1] + 2\delta[n-2] - 2\delta[n-2] - \delta[n-3]$$

$$= 2\delta[n] + 5\delta[n-1] - \delta[n-3]$$

Convolution sum evaluation procedure

Let $w_n[k] = x[k]h[n-k]$. Then $y[n]$ is expressed as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} w_n[k]$$

1. Graph both $x[k]$ and $h[k]$

2. Time reversal $h[k] \longrightarrow h[-k]$

3. Time shift $h[-k]$ by n shifts $\longrightarrow h[n-k]$ (left shift)

4. For a specific n, form product $x[k]h[n-k]$

5. Sum all samples of $x[k]h[n-k] \longrightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution sum evaluation procedure (cont.)

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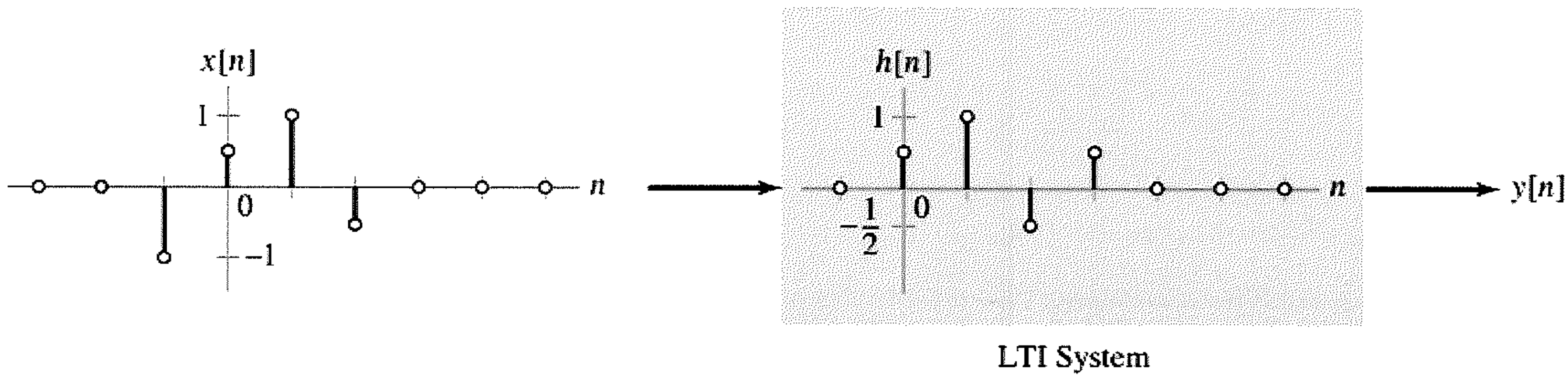
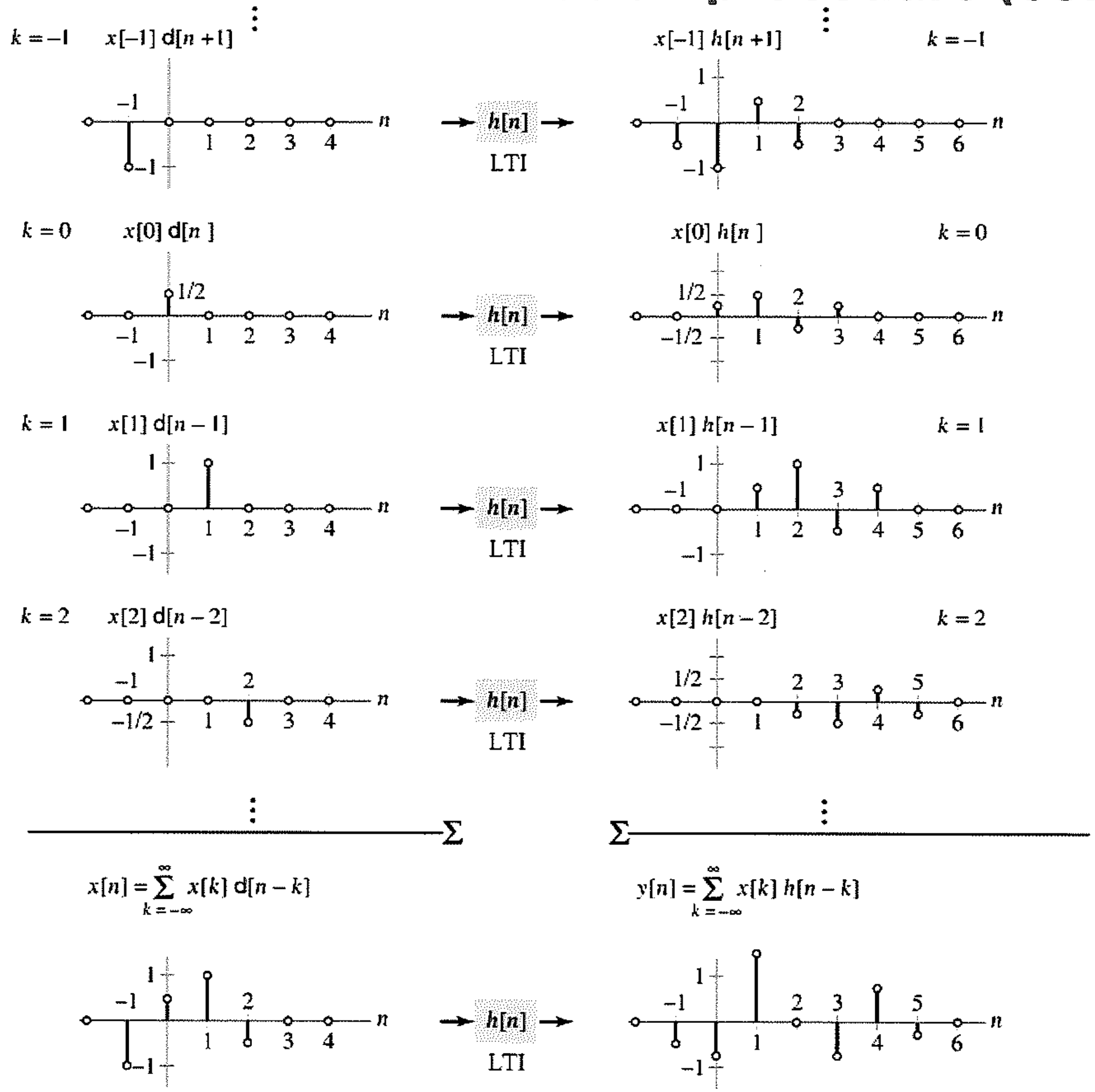
Graphical illustration of the convolution sum: (a) LTI system with impulse response $h[n]$ and input $x[n]$.



(a)

The decomposition of the input $x[n]$ into a weighted sum of time-shifted impulses results in an output $y[n]$ given by a weighted sum of time-shifted impulse responses next slide.

Convolution sum evaluation procedure (cont.)



Convolution sum evaluation procedure (cont.)

E: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ is applied to an LTI system with impulse response
$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$. Find $y[n]$.

$$y[n] = x[n] * h[n]$$

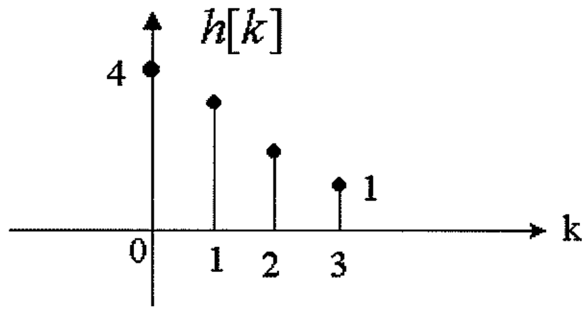
Exercise:

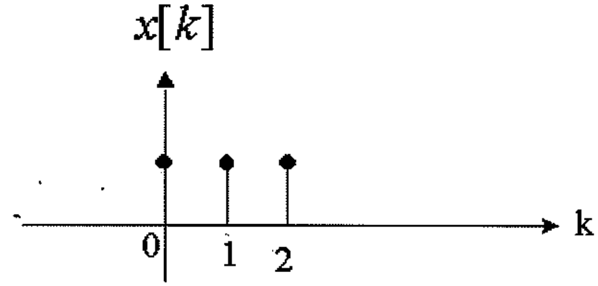
- Verify this the results using Matlab.

$$\begin{aligned} x &= [1 \;\; 1 \;\; 1]; \\ h &= [4 \;\; 3 \;\; 2 \;\; 1]; \\ y &= conv(x, h); \end{aligned}$$

Convolution sum evaluation procedure (cont.)

1)

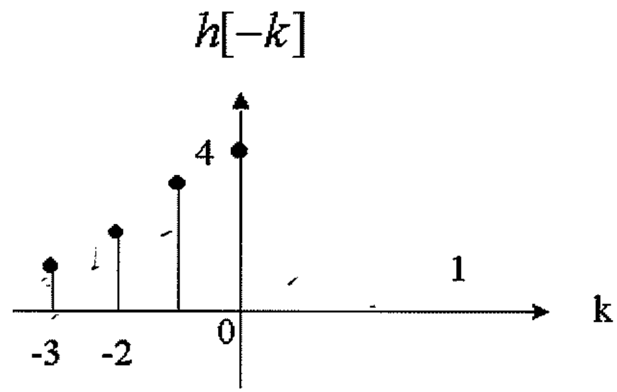




- Graph x[k] & h[k]

2) $h[-k]$

- Form h[-k] & time shift h[-k]



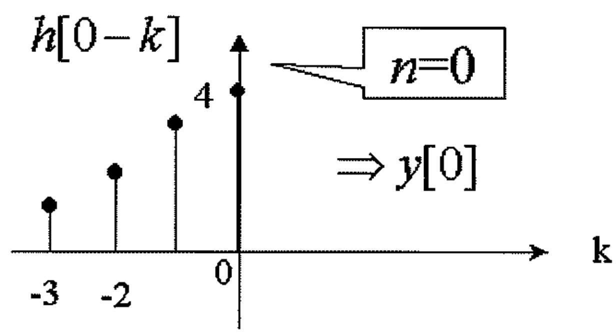
- For a specific n, form product

$$x[k]h[n-k] \Rightarrow \quad ...0 \quad 0 \quad 0 \quad 4 \quad 0 \; ... \quad \Rightarrow y[0] = \sum \; = 4$$

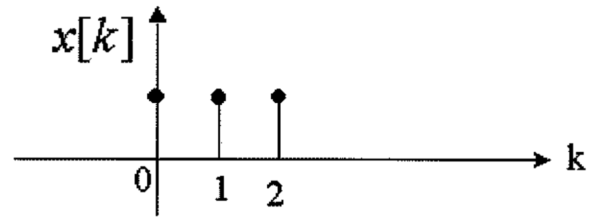
3) $h[n-k]$

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0 \quad \text{for } n < 0$$

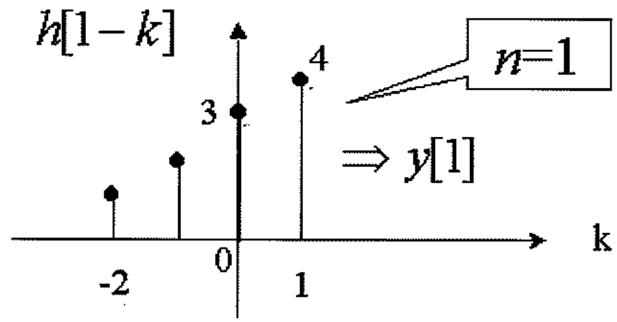
- Sum all products of $x[k]h[n-k]$



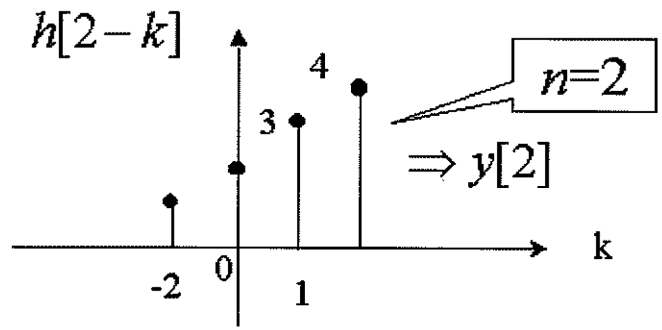
Convolution sum evaluation procedure (cont.)



$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 3 \quad 4 \quad 0 \quad 0 \ldots \Rightarrow y[1] = \sum \; = 7$



$x[k]h[n-k] \Rightarrow \quad \ldots 0 \quad 0 \quad 0 \quad 2 \quad 3 \quad 4 \quad 0 \ldots \Rightarrow y[2] = \sum \; = 9$



$y[4] = 3$
$y[5] = 1$
$y[n] = 0$ for $n \geq 6$

$\ldots 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 0 \ldots \Rightarrow y[3] = \sum \; = 6$

$y[n] =$ write in the form of $\sum \delta[n-k]$ yourselves.