Symmetry:

We will develop using continuous, non-periodic signals. Results for other 2 cases may be obtained in a similar way.

a) Assume $x(t)$ real $\Rightarrow x^*(t) = x(t)$

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}\,dt$$

$$= X(-j\omega)$$

$$\boxed{\text{If } x(t) \text{ is real} \Rightarrow X(j\omega) \text{ is conjugate symmetric}}$$

√ Further assume $x(t)$ even (and real)

$$x(-t) = x(t),\ x^*(t) = x(t) \Rightarrow x^*(t) = x(-t)$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega(-t)}dt$$

replace τ with $-t$ $= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$ (why?)

$$= X(j\omega)$$

$$\boxed{\begin{array}{l} \int_a^b x(t)dt = -\int_b^a x(t)dt \\ \tau = -t \Rightarrow d\tau = -dt \end{array}}$$

$\boxed{\text{If } x(t) \text{ is real and even} \Rightarrow X(j\omega) \text{ is real.}}$

√ Further assume $x(t)$ odd (and real)

$$x(-t) = -x(t),\ x^*(t) = x(t) \Rightarrow x^*(t) = -x(-t)$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} (-x(-t))e^{-j\omega(-t)}dt$$

$$= -X(j\omega)$$

$\boxed{\text{If } x(t) \text{ is real and odd} \Rightarrow X(j\omega) \text{ is purely imaginary.}}$

b) Assume $x(t)$ imaginary $\Rightarrow x^*(t) = -x(t)$

$$X^*(j\omega) = -\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt$$
$$= -X(-j\omega)$$

> If $x(t)$ is purely imaginary $\Rightarrow$
> - Real part of $X(j\omega)$ has odd symmetry
> - Imaginary part of $X(j\omega)$ has even symmetry

- <u>Convolution:</u> Applied to non-periodic signals.

Let $y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

If $x(t) \xleftrightarrow{FT} X(j\omega) \Rightarrow$

$$x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}d\omega$$

- <u>Convolution:</u> Applied to non-periodic signals.

$$\boxed{y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)}$$

Proof: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \Rightarrow x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}\,d\omega$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)\,H(j\omega)}\,e^{j\omega t}\,d\omega$$

$\cancel{Y(j\omega)} =$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}\,d\omega$$

$\Rightarrow Y(j\omega) = X(j\omega)\,H(j\omega)$

E

sinc t

Let $x(t) = \frac{1}{\pi t}\sin(\pi t)$ be input to a system with impulse response $h(t) = \frac{1}{\pi t}\sin(2\pi t)$. Find the system output $y(t)$

Solution:

$y(t) = x(t) * h(t)$

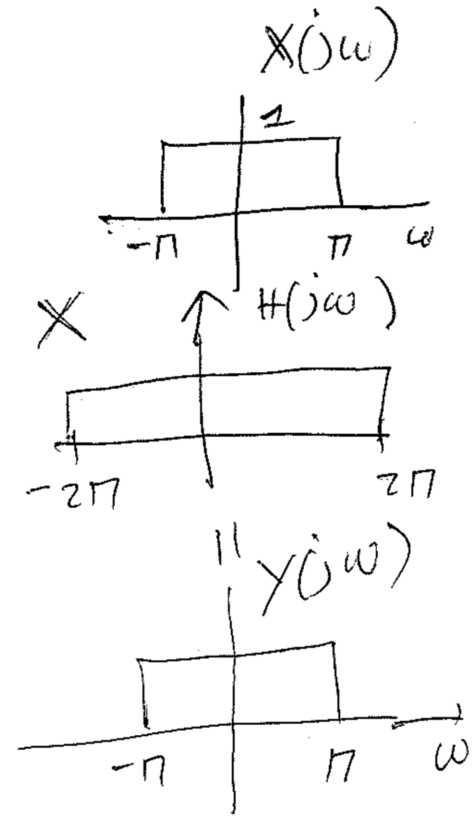
$x(t) \longleftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & \text{o.w} \end{cases}$

$h(t) \longleftrightarrow H(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & \text{o.w} \end{cases}$

$Y(j\omega) = X(j\omega)H(j\omega)$

$= X(j\omega)$

$y(t) = x(t) = \frac{1}{\pi t}\sin(\pi t)$



E

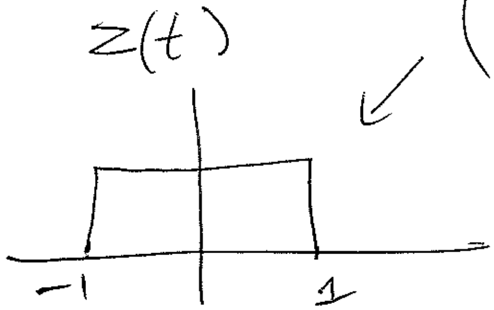
$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega)$. Find $x(t)$.

Solution: let $Z(j\omega) = \frac{2}{\omega} \sin \omega$

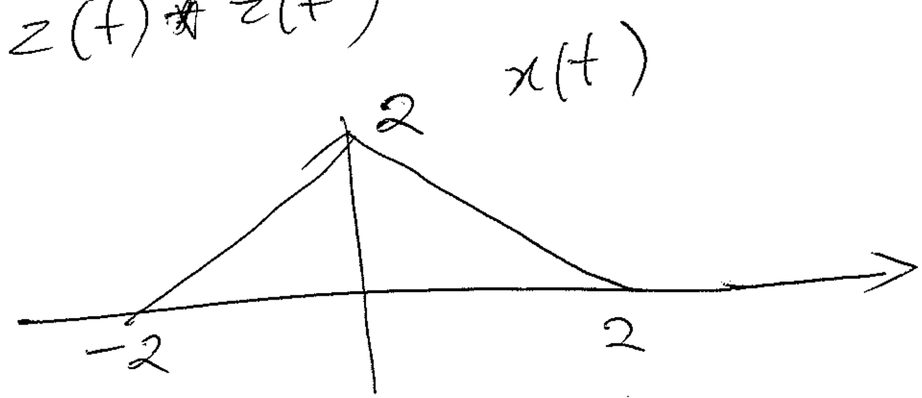
Note that $X(j\omega) = Z(j\omega)\, Z(j\omega)$

$x(t) = z(t) * z(t)$

(look it up from table)



$\cancel{x}(t) = z(t) * z(t)$



The same convolution properties hold for discrete-time, non-periodic signals.

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

- <u>Differentiation and integration:</u>
  - Applicable to continuous functions: time (t) or frequency (ω or Ω)
  - FT (t, ω) and DFTF (Ω)

<u>Differentiation in time:</u> $$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)}$$

Proof: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$

$$x'(t) = \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[j\omega X(j\omega)\right]e^{j\omega t}\,d\omega$$

$$\Downarrow$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$x'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X'(j\omega)e^{j\omega t}\,d\omega$$

$\boxed{\text{E}}$ Find FT of $\frac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0$

Solution:

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

From the table.

$$\frac{d\left(e^{-at}u(t)\right)}{dt} \xleftrightarrow{FT} \frac{j\omega}{a+j\omega}$$