Fourier representations of four classes of signals

Time property	Periodic	Nonperiodic
Continuous time (t)	• Fourier Series (FS) $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T}$ $X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt,$ (T: period)	• Fourier Transform (FT) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Discrete time [n]	• Discrete-Time Fourier Series (DTFS) $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}, \Omega_0 = \frac{2\pi}{N}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$ (N: period)	• Discrete-Time Fourier Transform (DTFT) $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

DTFS: *x[n]* periodic with period *N*, fundamental freq. $\Omega_0 = 2\pi/N$
DTFS coefficients of *x[n]: X[k].* Then

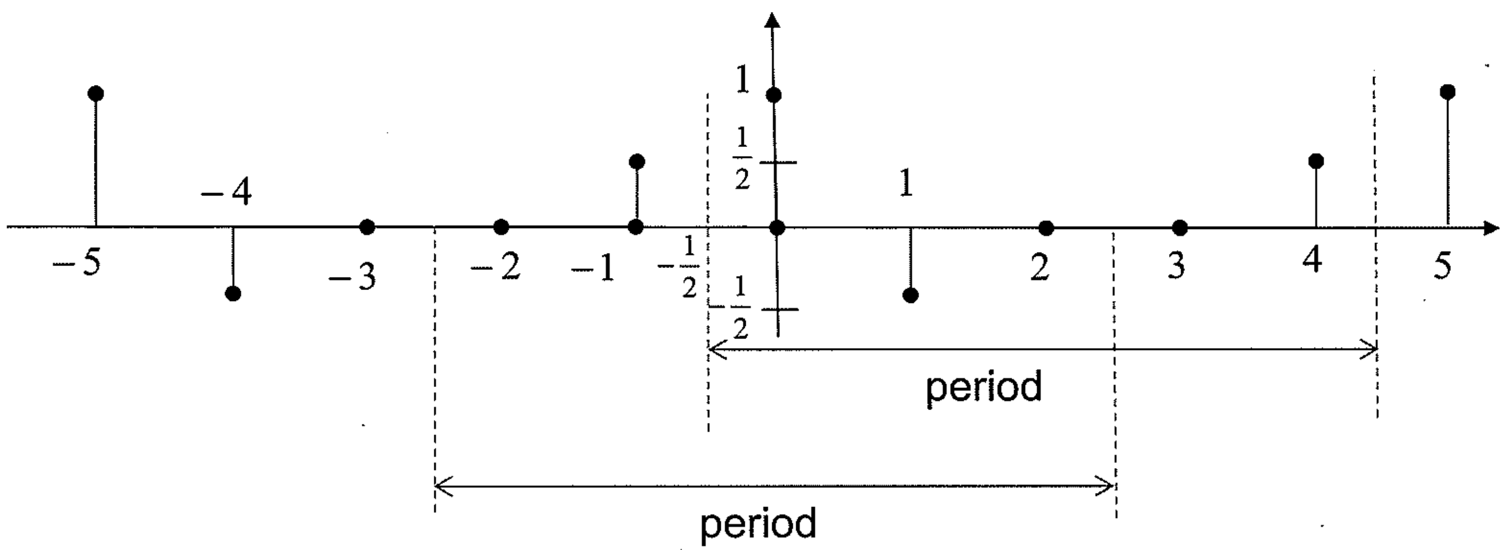
$$\begin{cases} x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \\ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} \end{cases}$$

Freq-domain representation of x[n]

x[n] and X[k] are a DTFS pair: $\boxed{x[n] \xleftrightarrow{DTFS;\, \Omega_0} X[k]}$

Note: a). Either x[n] or X[k] provides a complete description of the signal.
b). The limits on sums of x[n] or X[k] may be chosen differently from 0 to N-1.

E Find the freq-domain representation of x[n] given by



<u>Solution</u>: $N = 5$ (Period)

$\Omega_0 = 2\pi/N = 2\pi/5$ (Fundamental frequency)

1) $X[k] = ?$

There are two methods: a) From definition
b) Inspection

For both methods, it is important that we determine the the fundamental period correctly **first**.

a) By definition method:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} \quad , \quad N = 5, \quad \Omega_0 = \frac{2\pi}{5}$$

$$= \frac{1}{5}\sum_{n=0}^{4} x[n]e^{-jk\frac{2\pi}{5}n} = \frac{1}{5}\Big((1)e^{-jk\frac{2\pi}{5}(0)} + \left(-\tfrac{1}{2}\right)e^{-jk\frac{2\pi}{5}(1)} + (0)e^{-jk\frac{2\pi}{5}(2)} + (0)\left(e^{-jk\frac{2\pi}{5}(3)}\right) + \left(\tfrac{1}{2}\right)e^{-jk\frac{2\pi}{5}(4)}\Big)$$

$$= \frac{1}{5}\left(1 - \frac{1}{2}e^{-jk\frac{2\pi}{5}} + \frac{1}{2}e^{-jk\frac{2\pi 4}{5}}\right) \quad \square$$

$x[n] = x[n+N]$ N: fundamental period.

$X[k] = X[k+N]$

$$X[k+N] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(k+N)\Omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} e^{-jN\Omega_0 n}$$

$$= \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}}_{X[k]} \underbrace{e^{-j\left(\frac{2\pi}{\Omega_0}\right)\Omega_0 n}}_{=1} = X[k]$$

2) $x[n] = \cos\left(\frac{\pi}{3}n + \phi\right)$, $X[k] = ?$

$\Omega_0 = \pi/3$

$N = \frac{2\pi}{\pi/3} = 6$

b) Inspection method

$$x[n] = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2}$$ (Euler's formula)

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{j\Omega_0 kn} = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2}$$

3

$$\cancel{X[\text{odd}]} \quad \sum_{k=0}^{N-1} X[k]\, e^{j\frac{\pi}{3}kn} = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2}$$

$$= \frac{1}{2}\left(e^{j\phi}\right)\left(e^{j\frac{\pi}{3}n}\right) + \frac{1}{2}\left(e^{-j\phi}\right) e^{-j\frac{\pi}{3}n}$$

$$\Rightarrow \boxed{\begin{aligned} X[1] &= \tfrac{1}{2}e^{j\phi} \\ X[-1] &= \tfrac{1}{2}e^{-j\phi} \\ X[k] &= 0 \quad k \neq 1, -1 \quad \text{within the period} \end{aligned}}$$

$$X[7] = X[1]$$

$$X[5] = X[-1+6] = X[-1]$$