$\boxed{\text{E}}$ Find FT of $\frac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0$

$x(t) \longleftrightarrow X(j\omega)$

$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(j\omega)$

Solution:

E Find x(t) if $X(j\omega) = \begin{cases} j\omega, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$

Solution:

$$Z(j\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases} \longleftrightarrow z(t) = \frac{\sin t}{\pi t}$$

$$X(j\omega) = j\omega Z(j\omega)$$

$$x(t) = \frac{d z(t)}{dt}$$

$$z(t) = \frac{\sin t}{\pi t}$$

$$\boxed{x(t) = \frac{\cos t}{\pi t} - \frac{1}{(\pi t)^2} \sin t}$$

If x(t) is periodic, frequency-domain representation is Fourier Series (FS):

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$\boxed{\frac{d}{dt} x(t) \xleftrightarrow{FS;\omega_0} jk\omega_0 X[k]}$$

$$x(t) \longleftrightarrow X(j\omega)$$

Differentiation in frequency:

$$\boxed{-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)}$$

Proof:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} \underbrace{x(t)(-jt)} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \underbrace{[-jt\, x(t)]}_{\Updownarrow FT\ \frac{dX(j\omega)}{d\omega}} e^{-j\omega t} dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$\boxed{\text{E}}$

A Gaussian pulse is given as : $g(t)=\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$. Find its FT.

Solution:

$$x(t) \longleftrightarrow X(j\omega)$$

Integration:

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)}$$

E Determine the Fourier transform of u(t).

Solution:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

$$\delta(t) \longleftrightarrow 1.$$

$$U(j\omega) = \frac{1}{j\omega} + \pi(1)\,\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

E Find x(t), given

$$X(j\omega) = \frac{1}{j\omega(j\omega+1)} + \pi\delta(\cancel{j}\omega)$$

Solution:

$$X(j\omega) = \frac{1}{j\omega} - \frac{1}{j\omega+1} + \pi\delta(\omega)$$

$$= \left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{j\omega+1}$$

$$\Updownarrow \text{FT} \qquad\qquad \Updownarrow \text{FT}$$

$$u(t) - e^{-\cancel{a}t}u(t)$$

$$\boxed{\frac{1}{j\omega+a} \Longleftrightarrow e^{-at}u(t)}$$

$\boxed{\text{E}}$

$$x(t) = \frac{d}{dt}\left(2te^{-2t}u(t)\right) \qquad X(j\omega) = ?$$

- $e^{-2t}u(t) \xleftrightarrow{FT} \dfrac{1}{2+j\omega}$
- $te^{-2t}u(t) \xleftrightarrow{FT} \dfrac{1}{(2+j\omega)^2}$
- $\dfrac{d}{dt}\left(2te^{-2t}u(t)\right) \xleftrightarrow{FT} \dfrac{2j\omega}{(2+j\omega)^2}$

$$\boxed{-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)}$$

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)}$$

$$\frac{d\frac{1}{(2+j\omega)}}{d\omega} = -\frac{j}{(2+j\omega)^2} \Longleftrightarrow -jt\,x(t)$$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$z(t) = x(t-t_0) \longleftrightarrow Z(j\omega)\,?$$

- <u>Time and frequency shift</u>

<u>Time shift:</u> $\boxed{Z(j\omega) = e^{-j\omega t_0} X(j\omega)}$

Proof: $Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t}\, dt$$

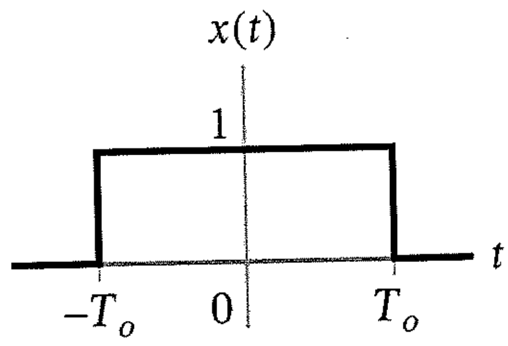
$\tau = t - t_0$ $\qquad = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)}\, d\tau$

$$= \left(\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau\right) e^{-j\omega t_0} = \left(X(j\omega)\right) e^{-j\omega t_0}$$

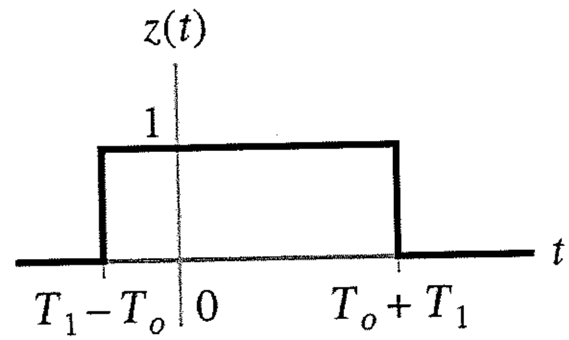
<u>Note:</u> Time shift $\Rightarrow$ phase shift in frequency domain. Phase shift is a linear function of ω. Magnitude spectrum does not change.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$
$$x(t-t_0) \xleftrightarrow{FS;\omega_0} e^{-jk\omega_0 t_0} X[k]$$
$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$
$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega_0 n_0} X[k]$$

E Find Z(jω)



(a)



(b)

$$z(t) = x(t-T_1)$$
$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$
$$z(t) \xleftrightarrow{FT} Z(j\omega) = e^{-j\omega T_1}\frac{2}{\omega}\sin(\omega T_0)$$

$\boxed{\text{E}}$ $X(j\omega) = \dfrac{e^{j4\omega}}{(2+j\omega)^2}$. Find $x(t)$

Solution:

$$\frac{1}{2+j\omega} \xleftrightarrow{FT} e^{-2t}u(t)$$

$$\frac{d\left(\frac{1}{2+j\omega}\right)}{d\omega}$$

$$-\frac{j}{(2+j\omega)^2} \xleftrightarrow{FT} -jte^{-2t}u(t)$$

$$\frac{1}{(2+j\omega)^2} \xleftrightarrow{FT} te^{-2t}u(t)$$

$$\frac{e^{j4\omega}}{(2+j\omega)^2} \xleftrightarrow{F} (t+4)e^{-2(t+4)}u(t+4)$$

Frequency shift:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$X(j(\omega-\gamma)) \xleftrightarrow{FT} e^{j\gamma t}x(t)$$

Proof:

you can try this yourself.

Note:
- Frequency shift $\Rightarrow$ time signal multiplied by a complex sinusoid.
- Carrier modulation.

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$
$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\omega_0} X[k-k_0]$$
$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$
$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{DTFS;\Omega_0} X[k-k_0]$$

$\boxed{E}$

$$z(t) = \begin{cases} e^{j10t}, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$$

Find Z(jω).

$$x(t) = \begin{cases} 1 & |t| < \pi \\ 0 & |t| > \pi \end{cases}$$

$$X(j\omega) = \frac{2\sin\omega\pi}{\omega}$$

Solution:

$$z(t) = e^{j10t}\, x(t)$$

$$Z(j\omega) = X(j(\omega - 10))$$

$$= \frac{2 \sin(\omega - 10)\pi}{(\omega - 10)}$$

$\boxed{\text{E}}$ $x(t) = \frac{d}{dt}\left\{\left(e^{-3t}u(t)\right) * \left(e^{-t}u(t-2)\right)\right\}$. Find $X(j\omega)$.

Solution:

$$e^{-3t}u(t) \xleftrightarrow{FT} \frac{1}{3+j\omega}$$

$$e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega}$$

$$e^{-(t-2)}u(t-2) \longleftrightarrow \frac{e^{-2\omega}}{1+j\omega}$$

$$e^{-t}u(t-2) \longleftrightarrow \frac{1}{e^2}\frac{e^{-j2\omega}}{1+j\omega}$$

$$\left(e^{-3t}u(t)\right) * \left(e^{-t}u(t-2)\right) \longleftrightarrow \frac{1}{e^2}\left(\frac{1}{3+j\omega}\right)\left(\frac{e^{-j2\omega}}{1+j\omega}\right)$$

$$\frac{d\left(e^{-3t}u(t) * e^{-t}u(t-2)\right)}{dt} \longleftrightarrow \frac{j\omega}{e^2}\left(\frac{1}{3+j\omega}\right)\left(\frac{e^{-j2\omega}}{1+j\omega}\right)$$