- Multiplication

CT, non-periodic

$$\boxed{y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} X(j\omega) * Z(j\omega)}$$

DT, non-periodic

$$\boxed{y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})}$$

Periodic convolution: $X(e^{j\Omega}) \circledast Z(e^{j\Omega}) = \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Z\left(e^{j(\Omega-\theta)}\right) d\theta$

CT, periodic

$$\boxed{y(t) = x(t)z(t) \xleftrightarrow{FS;2\pi/T} Y[k] = X[k] * Z[k]}$$

DT, periodic

$$y[n] = x[n]z[n] \xleftrightarrow{DTFS;2\pi/N} Y[k] = X[k] \circledast Z[k]$$

$$X[k] \circledast Z[k] = \sum_{m=0}^{N-1} X[m] Z[k-m]$$

- <u>Scaling</u> $x(at) \leftrightarrow \frac{1}{|a|} X(j(\omega / a))$

Proof: $z(t) = x(at)$

$$Z(j\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

$$\tau = at$$

$$dt = \frac{d\tau}{a} \quad = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau, & a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau, & a < 0 \end{cases}$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

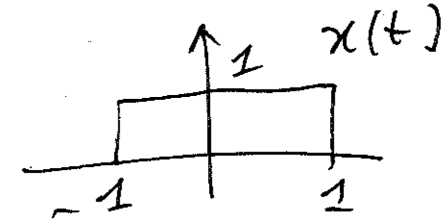
$$= \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$$

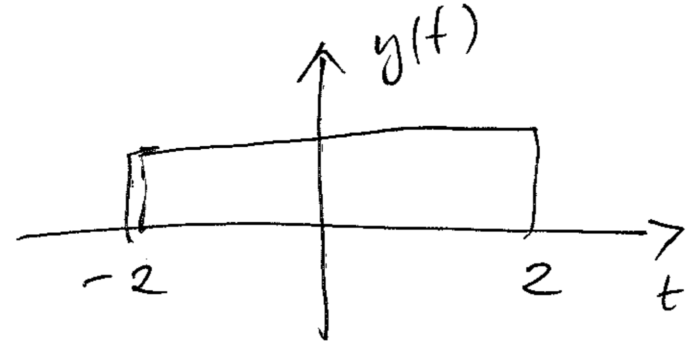
E

$$x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

$$y(t) = \begin{cases} 1 & |t| < 2 \\ 0 & |t| > 2 \end{cases}$$

Find $Y(j\omega)$





Solution:

$$y(t) = x\left(\tfrac{1}{2} t\right)$$

$$x(t) \overset{F}{\longleftrightarrow} \frac{2\sin\omega}{\omega} = X(j\omega)$$

$$y(t) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right) = \frac{1}{\left(\frac{1}{2}\right)} \frac{2\sin 2\omega}{2\omega}$$

$$= \frac{2\sin 2\omega}{\omega}$$

E

Find $x(t)$ if $X(j\omega) = j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

We know $s(t) = e^{-t}u(t) \xleftrightarrow{FT} S(j\omega) = \dfrac{1}{1+j\omega}$

– Time scaling: $z(t) = s(3t) \xleftrightarrow{FT} Z(j\omega) = \dfrac{1}{3}\dfrac{1}{(1+j(\omega/3))}$

– Time shift: $v(t) = 3z(t+2) = 3s(3(t+2)) \xleftrightarrow{FT} \dfrac{e^{j2\omega}}{1+j(\omega/3)}$

–Differentiation: $tv(t) \xleftrightarrow{FT} j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|}X(j(\omega/a))$$
$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0}X(j\omega)$$
$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

Thus,
$$\begin{aligned} x(t) &= tv(t) \\ &= 3tz(t+2) \\ &= 3ts(3(t+2)) \\ &= 3te^{-3(t+2)}u(3(t+2)) \end{aligned}$$

Note: $u(3(t+2)) = u(t+2)$. Thus, $x(t) = 3te^{-3(t+2)}u(3(t+2))$

- <u>Parseval's relationship:</u>

Energy of CT, non-periodic signal $x(t)$: $W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$= \int_{-\infty}^{\infty} x^*(t)x(t)dt$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

$$W_x = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right\}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)d\omega$$

$$\boxed{W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

Note: a) $|X(j\omega)|^2$: energy spectrum

b) Energy in time domain = energy in freq. domain

- DTFT: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$
- DTFS: $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$
- FS: $\frac{1}{T}\int_0^T |x(x)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$

E

$x[n] = \frac{\sin(Wn)}{\pi n}$. Determine $E_x = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{(\pi n)^2}$

$$x[n] = \frac{\sin(Wn)}{\pi n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| \le \pi \end{cases}$$

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega$$

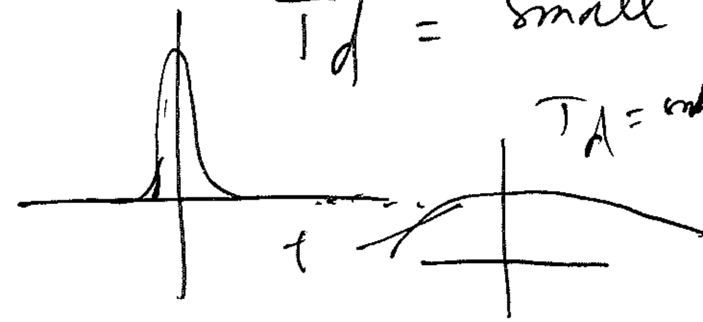
$$= \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = W / \pi$$

- Time-bandwidth product

Compression in time domain $\Rightarrow$ expansion in frequency domain

Bandwidth: The extent of the signal's significant contents. It is in general a vague definition as "significant" is not mathematically defined. In practice, definitions of bandwidth include

T_d = small



T_A =

– absolute bandwidth
– x% bandwidth
– first-null bandwidth.

If we define

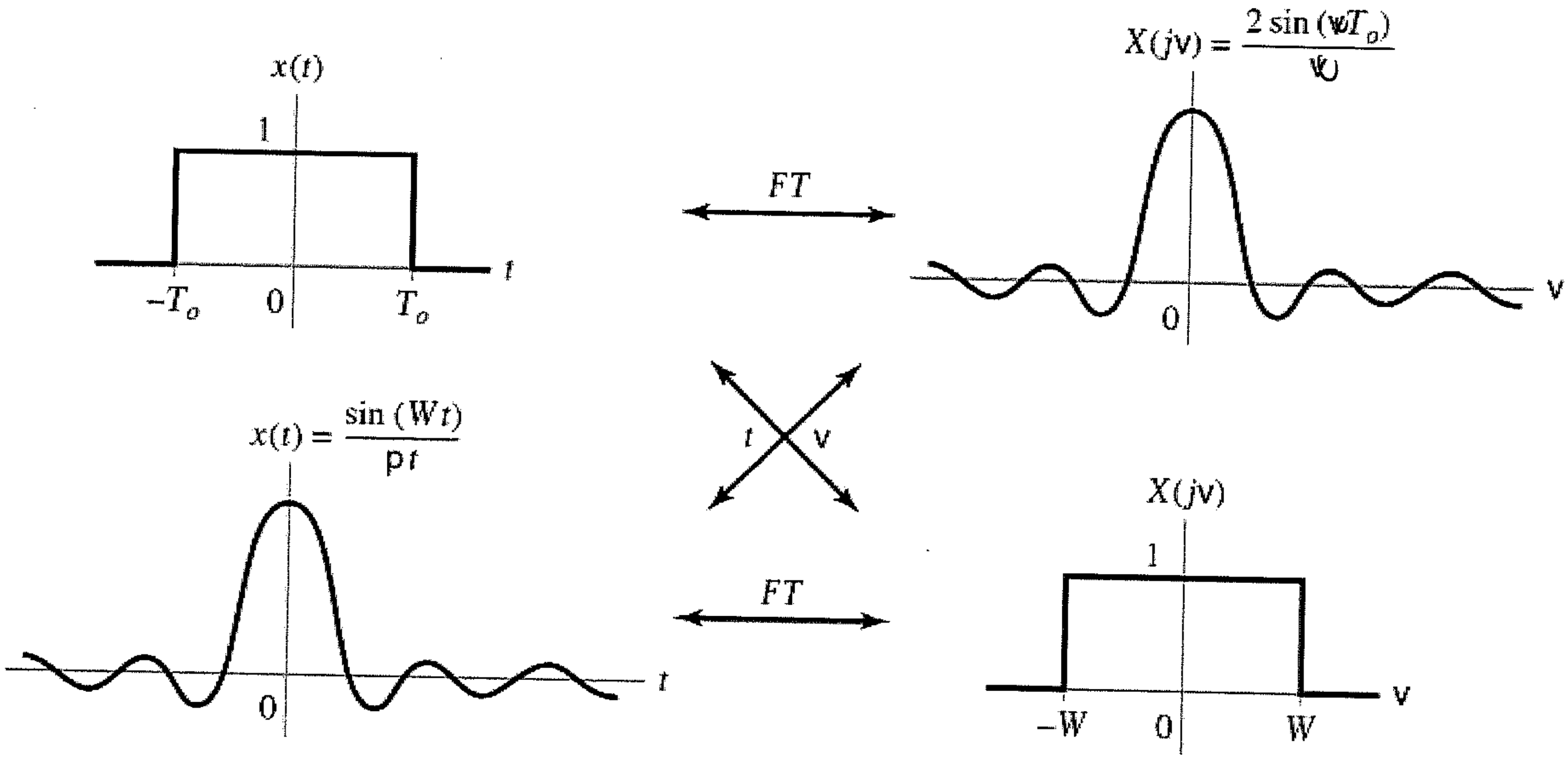
$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2\, dt}{\int_{-\infty}^{\infty} |x(t)|^2\, dt}\right]^{1/2} \quad : \text{ RMS duration of an energy signal}$$

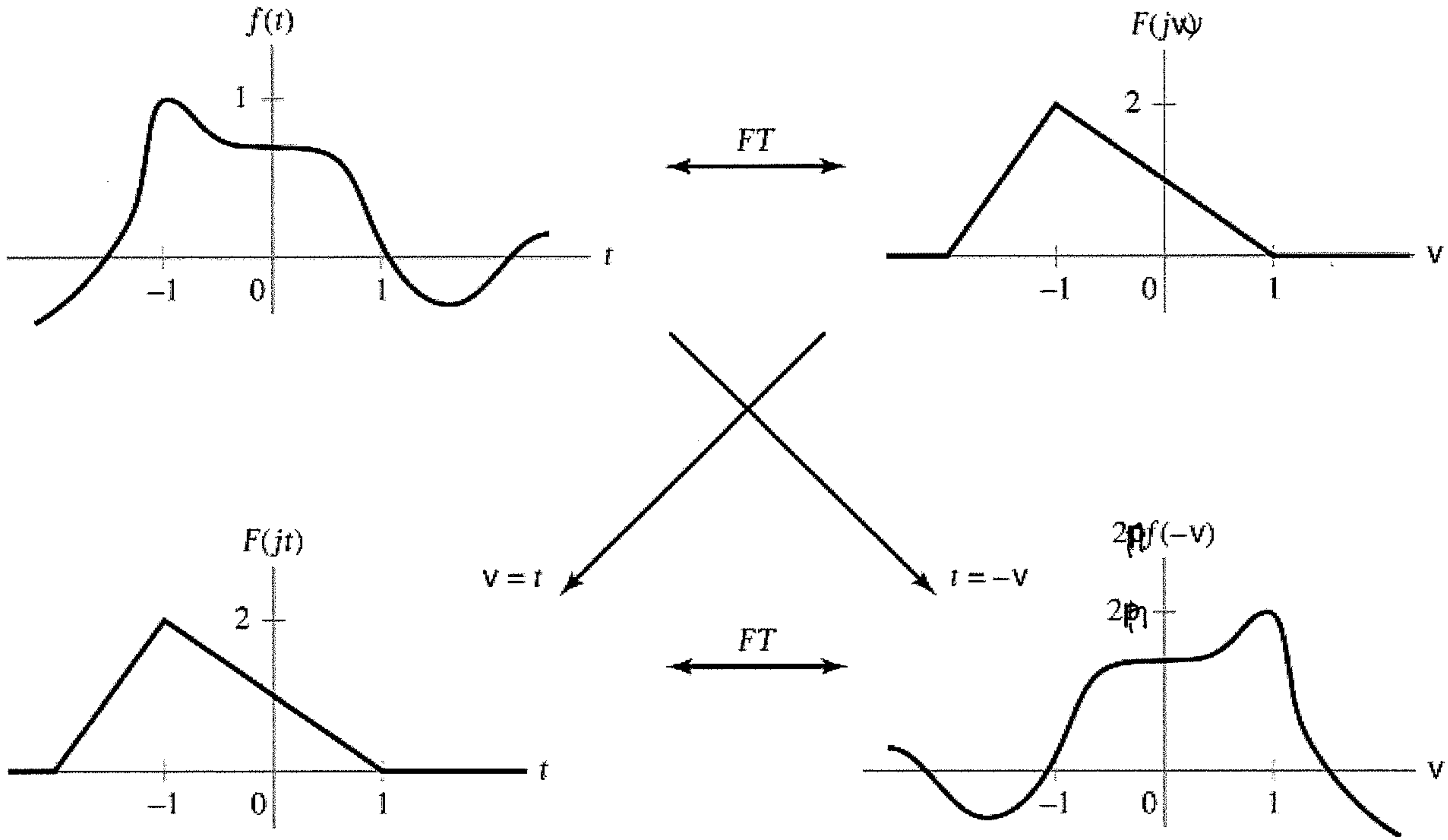
$$B_w = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2\, d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega}\right]^{1/2} \quad : \text{ RMS bandwidth, then}$$

$$\boxed{T_d B_w \geq 1/2}$$

- <u>Duality</u>

$$\boxed{\begin{array}{l} f(t) \xleftrightarrow{FT} F(j\omega) \\ F(t) \xleftrightarrow{FT} 2\pi f(-\omega) \end{array}}$$





f(t)
1
–1
0
1
t
FT
F(jv)
2
–1
0
1
v
v = t
t = –v
F(jt)
2
–1
0
1
t
FT
2πf(–v)
2π
–1
0
1
v

$\boxed{\text{E}}$ Find $X(j\omega)$ if $x(t) = \dfrac{1}{1+jt}$

We know $f(t) = e^{-t}u(t) \xleftrightarrow{FT} \dfrac{1}{1+j\omega} = F(j\omega)$

Duality $F(jt) = \dfrac{1}{1+jt} \xleftrightarrow{FT} 2\pi f(-\omega) \Rightarrow$

$$X(j\omega) = 2\pi f(-\omega) = 2\pi e^{\omega} u(-\omega)$$

Check $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(j\omega) e^{-j\omega t} d\omega$

$= \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0} 2\pi e^{\omega} e^{j\omega t} d\omega$

$= \displaystyle\int_{-\infty}^{0} e^{\omega(1+jt)} d\omega = \dfrac{1}{1+jt}$

$\boxed{\text{E}}$ Find $x(t)$ if ~~$X(j\omega) = \omega u(\omega)$~~

$X(j\omega) = u(\omega)$

$$X(j\omega) = u(\omega)$$

$$X(jt) = u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(+\omega) = 2\pi x(-\omega) \Rightarrow$$

$$x(t) = \left[\frac{1}{j(-t)} + \pi\delta(-(-t))\right]\frac{1}{2\pi}$$

$$= \frac{-1}{2\pi jt} + \frac{1}{2}\delta(t)$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)}$$

- DTFS: $\begin{cases} x[n] \xleftrightarrow{DTFS;2\pi/N} X[k] \\ X[n] \xleftrightarrow{DTFS;2\pi/N} \frac{1}{N}x[-k] \end{cases}$
- DTFT and FS: $\begin{cases} x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) \\ X\left(e^{jt}\right) \xleftrightarrow{FS;1} x[-k] \end{cases}$
- *D*TFT and FS do not stay in their own class!