Can you have more than one Nash Equilibrium?

• ACME, a video game hardware manufacturer, has to decide whether its next game machine will use DVDs or CDs
• Best, a video game software producer, needs to decide whether to produce its next game on DVD or CD
• Profits for both will be positive if they agree and negative if they disagree

There are two Nash Equilibria in this game. In general, you can multiple Nash Equilibria. This creates a big problem. Can you see what that problem is?

Dealing with Multiple Nash Equilibria

1. Could choose the Pareto-optimal Nash Equilibrium eg. (dvd, dvd) but
   - What if there are multiple Pareto-optimal Nash Equilibria?
   - Or it’s too computationally expensive to find all the Nash Equilibria?
   - Or there are an infinite number of Nash Equilibria?
2. Could communicate before the game
   - But what if you can’t compute all the Nash Equilibria beforehand?
3. Take your best guess

This is a big unresolved issue
Two Fingered Morra

<table>
<thead>
<tr>
<th></th>
<th>O: one</th>
<th>O: two</th>
</tr>
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<tbody>
<tr>
<td>E: one</td>
<td>E = 2, O = -2</td>
<td>E = -3, O = 3</td>
</tr>
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<td>E: two</td>
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<td>E = 4, O = -4</td>
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- No pure strategy Nash Equilibrium
- If total # of fingers is even, O will want to switch
- If total # of fingers is odd, E will want to switch
- Also, this is a zero-sum game (payoffs in a cell sum to zero)

The Big Theorem

- [Nash 1950] In the n-player normal-form game $G=\{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if $n$ is finite and $S_i$ is finite for every $i$ then there exists at least one Nash Equilibrium, possibly involving mixed strategies

Mixed Strategies

- Recall that a pure strategy is a deterministic policy ie. you pick a strategy and play it all the time
- A mixed strategy is a randomized policy ie. you select your strategy based on a probability distribution
- eg. Select strategy $S_1$ with probability $p$ and strategy $S_2$ with probability $(1-p)$
- Is there a mixed strategy Nash Equilibrium in 2 Fingered Morra?

Formal Definition of a Mixed Strategy

In the normal-form game $G=\{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, suppose $S_i = \{s_{i1}, \ldots, s_{ik}\}$. Then a mixed strategy for a player $i$ is a probability distribution $p_i = (p_{i1}, \ldots, p_{ik})$.

$$0 \leq p_{ik} \leq 1 \text{ for } k = 1, \ldots, K$$

and $p_{i1} + \ldots + p_{ik} = 1$.

Mixed Strategy Nash Equilibrium

- The pair of mixed strategies $(M_A, M_B)$ are a Nash Equilibrium iff
- Player A does not want to deviate from $M_A$ (because $M_A$ is Player A’s best response to $M_B$ and)
- Player B does not want to deviate from $M_B$ (because $M_B$ is Player B’s best response to $M_A$)

Finding optimal mixed strategy for two-player zero-sum games

- Note: applies to zero-sum games (or, more generally, constant sum games)
- Von Neumann’s maximin technique
Expected Payoff to E if O Uses a Mixed Strategy

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Suppose O chooses to display one finger with probability p and two fingers with probability (1-p).

If E chooses the pure strategy of one finger, E’s expected profit is 2p - 3(1-p) = 2p - 3 + 3p = 5p - 3.

If E chooses the pure strategy of two fingers, E’s expected profit is -3p + 4(1-p) = -3p + 4 - 4p = -7p + 4.

Expected Payoff to E if O Uses a Mixed Strategy

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Expected Payoff to E if E Uses a Mixed Strategy

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Suppose E chooses to display one finger with probability q and two fingers with probability (1-q).

If E chooses the pure strategy of one finger, O’s expected payoff is -2q + 3(1-q) = -2q + 3 - 3q = -5q + 3.

If E chooses the pure strategy of two fingers, O’s expected payoff is 3q - 4(1-q) = 3q - 4 + 4q = 7q - 4.

Expected Payoff to E if E Uses a Mixed Strategy

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Expected Payoff to O if E Uses a Mixed Strategy

\[5p - 3 = -7p + 4\]  
\[\Rightarrow 12p = 7\]  
\[\Rightarrow p = 7/12\]

When \(p < 7/12\), E plays ‘two’.

When \(p > 7/12\), E plays ‘one’.

O gets to pick \(p\) to minimize E’s expected payoff. O picks the lowest point of the higher of the two lines. This happens at the intersection of the two lines.

E’s expected payoff at \(p=7/12\) is \((5/7)(2)+3 = 1/12\).

O’s mixed strategy is \((7/12\text{ for ‘one’}, 5/12\text{ for ‘two’})\).

Mixed Strategy

- E’s expected payoff is -1/12, O’s is 1/12.
- It is better to be O than to be E.
- The final mixed strategy is for both players to play “one” with probability 7/12 and “two” with probability 5/12.
- This is a maximin equilibrium (which is also a Nash equilibrium).

Theoretical Results

- Every two-player zero-sum game has a maximin equilibrium when you allow mixed strategies.
- Every Nash equilibrium in a two-player zero-sum game is a maximin equilibrium for both players.

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Recipe for Computing Optimal Mixed Strategy 2x2 Constant-Sum Games

- Let Player B use strategy S1 with probability p
- Compute Player A’s expected payoff if A uses pure strategy S1: \( m_1p + m_2(1-p) \)
- Compute Player A’s expected payoff if A uses pure strategy S2: \( m_12p + m_22(1-p) \)
- Find the p between 0 and 1 that minimizes \( \max( m_11p + m_21(1-p), m_12p + m_22(1-p) ) \)
- The optimum will be at \( p=0, p=1 \) or at the point where the two lines intersect
- Repeat by letting Player A use Strategy S1 with probability q but looking at B’s payoffs now

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<td>A: S1</td>
<td>( m_1 )</td>
<td>( m_2 )</td>
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<td>A: S2</td>
<td>( m_12 )</td>
<td>( m_22 )</td>
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Recipe for Computing Optimal Mixed Strategy NxM Zero-Sum Games

- NxM game = Player A has N pure strategies, Player B has M pure strategies
- Let Player B use:
  - Strategy S1 with probability \( p_1 \)
  - Strategy S2 with probability \( p_2 \)
  - Strategy SN with probability \( p_N \)
- Compute Player A’s expected payoff if A uses:
  - Pure strategy S1: \( e_1 = m_11p_1 + m_21p_2 + \ldots + m_N1p_N \)
  - Pure strategy S2: \( e_2 = m_12p_1 + m_22p_2 + \ldots + m_N2p_N \)
  - Pure strategy SM: \( e_M = m_1Mp_1 + m_2Mp_2 + \ldots + m_NMp_N \)
- Find \( p_1, p_2, \ldots, p_N \) to minimizes \( \max( e_1, e_2, \ldots, e_M ) \) subject to \( \Sigma p_i = 1 \) and \( 0 \leq p_i \leq 1 \) for all i
- Use a method called Linear Programming (polynomial time in number of actions)
- Repeat by letting Player A use a mixed strategy and looking at Player B’s payoffs

What About Two-Player Non-Zero Sum Games?

- This is a linear complementarity problem
- Use the Lemke-Howson algorithm
- If interested, see “Computing Equilibria for Two-Person Games” by Bernhard von Stengel

Battle of the Sexes

- Traditional game from 1950s (this version is slightly modified and gender neutral)
- A man and a woman are trying to decide on what to do for entertainment. They are at separate workplaces and must decide whether to attend the Oprah Winfrey show or go to a football game
- Both players would rather spend time together than apart
- Player A would rather go to Oprah’s show while Player B would rather go to a football game.

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<tr>
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<th>B: Football</th>
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<td>( A = 2, B = 1 )</td>
<td>( A = 0, B = 0 )</td>
</tr>
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<td>A: Football</td>
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<td>( A = 1, B = 2 )</td>
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Suppose A plays Oprah with probability \( p \) and Football with probability \( 1-p \)

- B chooses the pure strategy of Oprah. Expected payoff for B: \( p \)
- B chooses the pure strategy of Football. Expected payoff for B: \( 2(1-p) \)
- \( p = 2(1-p) \)
- \( 3p = 2 \Rightarrow p = 2/3 \)
- Mixed Strategy for A (2/3,1/3), Expected payoff for B is 2/3
### Battle of the Sexes

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Suppose B plays Oprah with probability $q$ and football with probability $(1-q)$

- A chooses the pure strategy of Oprah. Expected payoff for A: $2q$
- A chooses the pure strategy of Football. Expected payoff for A: $1-q$

\[
2q = 1-q \\
\Rightarrow 3q = 1 \\
\Rightarrow q = \frac{1}{3}
\]

- Mixed Strategy for B $(1/3, 2/3)$, Expected payoff for A is $2/3$

### What you should know

- How to find pure strategy Nash Equilibria in a game
- Problems with having multiple Nash Equilibria
- How to compute mixed strategy Nash Equilibria in two-player constant sum games