Informed Search

• How can we make search smarter?
• Use problem-specific knowledge beyond the definition of the problem itself
• Specifically, incorporate knowledge of how good a non-goal state is
Best-First Search

• Node selected for expansion based on an evaluation function $f(n)$ ie. expand the node that appears to be the best
• Node with lowest evaluation is selected for expansion
• Uses a priority queue
• We’ll talk about Greedy Best-First Search and A* Search

Heuristic Function

• $h(n) =$ estimated cost of the cheapest path from node $n$ to a goal node
• $h($goal node$) = 0$
• Contains additional knowledge of the problem
Greedy Best-First Search

- Expands the node that is closest to the goal
- $f(n) = h(n)$

Greedy Best-First Search Example

<table>
<thead>
<tr>
<th>City</th>
<th>Distance to Wilsonville</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corvallis</td>
<td>56</td>
</tr>
<tr>
<td>Albany</td>
<td>49</td>
</tr>
<tr>
<td>Salem</td>
<td>28</td>
</tr>
<tr>
<td>Portland</td>
<td>17</td>
</tr>
<tr>
<td>McMinnville</td>
<td>18</td>
</tr>
</tbody>
</table>
Greedy Best-First Search Example
Greedy Best-First Search Example

Corvallis → McMinnville → Wilsonville = 74 miles

But the route below is much shorter than the route found by Greedy Best-First Search!

Corvallis → Albany → Salem → Wilsonville = 67 miles
Evaluating Greedy Best-First Search

<table>
<thead>
<tr>
<th>Complete?</th>
<th>No (could start down an infinite path)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal?</td>
<td>No</td>
</tr>
<tr>
<td>Time Complexity</td>
<td>$O(b^m)$</td>
</tr>
<tr>
<td>Space Complexity</td>
<td>$O(b^m)$</td>
</tr>
</tbody>
</table>

Greedy Best-First search results in lots of dead ends which leads to unnecessary nodes being expanded

A* Search

- A much better alternative to greedy best-first search
- Evaluation function for A* is:
  
  $$f(n) = g(n) + h(n)$$
  
  where $g(n)$ = path cost from the start node to $n$
- If $h(n)$ satisfies certain conditions, A* search is optimal and complete!
Admissible Heuristics

- A* is optimal if h(n) is an admissible heuristic
- An admissible heuristic is one that never overestimates the cost to reach the goal
- Admissible heuristic = optimistic
- Straight line distance was an admissible heuristic

A* Search Example

\[ f(n) = g(n) + h(n) \]
A* Search Example

Corvallis

McMinnville
46 + 18 = 64

Albany
11 + 49 = 60

Salem
37 + 28 = 65

Corvallis
22 + 56 = 78
A* Search Example

Note: Don’t stop when you put a goal state on the priority queue (otherwise you get a suboptimal solution)

Proper termination: Stop when you pop a goal state from the priority queue
Proof that A* using TREE-SEARCH is optimal if h(n) is admissible

- Suppose a suboptimal goal node $G_2$ appears on the fringe and let the cost of the optimal solution be $C^*$
- Because $G_2$ is suboptimal:
  \[ f(G_2) = g(G_2) + h(G_2) = g(G_2) > C^* \]
- Now consider a fringe node $n$ on an optimal solution path
- If $h(n)$ is admissible then:
  \[ f(n) = g(n) + h(n) \leq C^* \]
- We have shown that $f(n) \leq C^* < f(G_2)$, so $G_2$ will not get expanded and A* must return an optimal solution

What about search graphs (more than one path to a node)?

- What if we expand a state we’ve already seen?
- Suppose we use the GRAPH-SEARCH solution and not expand repeated nodes
- Could discard the optimal path if it’s not the first one generated
- One simple solution: ensure optimal path to any repeated state is always the first one followed (like in Uniform-cost search)
- Requires an extra requirement on $h(n)$ call consistency (or monotonicity)
Consistency

• A heuristic is consistent if, for every node \( n \) and every successor \( n' \) of \( n \) generated by any action \( a \):
  \[ h(n) \leq c(n,a,n') + h(n') \]
  
  Step cost of going from \( n \) to \( n' \)
  by doing action \( a \)

• A form of the triangle inequality – each side of the triangle cannot be longer than the sum of the two sides

Consistency

• Every consistent heuristic is also admissible
• A* using GRAPH-SEARCH is optimal if \( h(n) \) is consistent
• Most admissible heuristics are also consistent
**Consistency**

- If \( f(n) \) is consistent, then the values of \( f(n) \) along any path are nondecreasing
- **Proof:**
  
  Suppose \( n' \) is a successor of \( n \).
  
  Then \( g(n') = g(n) + c(n, a, n') \) for some \( a \), and we have
  
  \[
  f(n') = g(n') + h(n') \\
  = g(n) + c(n, a, n') + h(n') \\
  \geq g(n) + h(n) \\
  = f(n)
  \]

  From defn of consistency:
  
  \[
  c(n, a, n') + h(n') \geq h(n)
  \]

- Thus, the sequence of nodes expanded by A* is in nondecreasing order of \( f(n) \)
- First goal selected for expansion must be an optimal solution since all later nodes will be at least as expensive

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**A* is Optimally Efficient**

- Among optimal algorithms that expand search paths from the root, A* is *optimally efficient* for any given heuristic function
- Optimally efficient: no other optimal algorithm is guaranteed to expand fewer nodes than A*
  
  - Fine print: except A* might possibly expand more nodes with \( f(n) = C^* \) where \( C^* \) is the cost of the optimal path – tie-breaking issues
- Any algorithm that does not expand all nodes with \( f(n) < C^* \) runs the risk of missing the optimal solution
Evaluating A* Search

With a consistent heuristic, A* is complete, optimal and optimally efficient. Could this be the answer to our searching problems?

The Dark Side of A*…

Time complexity is exponential (although it can be reduced significantly with a good heuristic)

The really bad news: space complexity is exponential (usually need to store all generated states). Typically runs out of space on large-scale problems.
Summary of A* Search

| Complete? | Yes if h(n) is consistent, b is finite, and all step costs exceed some finite \( \varepsilon \) \(^1\) |
| Optimal? | Yes if h(n) is consistent and admissible |
| Time Complexity | \( O(b^d) \) (In the worst case but a good heuristic can reduce this significantly) |
| Space Complexity | \( O(b^d) \) – Needs \( O(\text{number of states}) \), will run out of memory for large search spaces |

\(^1\) Since \( f(n) \) is nondecreasing, we must eventually hit an \( f(n) = \text{cost of the path to a goal state} \)

Iterative Deepening A*

- Use iterative deepening trick to reduce memory requirements for A*
- Cutoff is the f-cost \((g+h)\) rather than the depth
- At each iteration, the cutoff is the smallest f-cost that exceeded the cutoff in the previous iteration

Complete, Optimal but more costly than A* and can take a while to run with real-valued costs
Examples of heuristic functions

The 8-puzzle

Heuristic #1: \( h_1 = \text{number of misplaced tiles} \) eg. start state has 8 misplaced tiles. This is an admissible heuristic.

Heuristic #2: \( h_2 = \text{total Manhattan distance (sum of horizontal and vertical moves, no diagonal moves)} \). Start state is 3+1+2+2+2+3+3+2=18 moves away from the end state. This is also an admissible heuristic.
Which heuristic is better?

- $h_2$ dominates $h_1$. That is, for any node $n$, $h_2(n) \geq h_1(n)$.
- $h_2$ never expands more nodes than $A^*$ using $h_1$ (except possibly for some nodes with $f(n) = C^*$).
- Better to use $h_2$ provided it doesn’t overestimate and its computation time isn’t too expensive.
  
  (Remember that $h_2$ is also admissible)

Proof:
Every node with $f(n) < C^*$ will surely be expanded, meaning every node with $h(n) < C^* - g(n)$ will surely be expanded.

Since $h_2$ is at least as big as $h_1$ for all nodes, every node expanded with $A^*$ using $h_2$ will also be expanded with $A^*$ using $h_1$. But $h_1$ might expand other nodes as well.

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<table>
<thead>
<tr>
<th>Depth</th>
<th>IDS</th>
<th>$A^*(h_1)$</th>
<th>$A^*(h_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>112</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>680</td>
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<tr>
<td>8</td>
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<tr>
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<td>12</td>
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<tr>
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<td>1641</td>
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</tr>
</tbody>
</table>

From Russell and Norvig Figure 4.8 (Results averaged over 100 instances of the 8-puzzle for depths 2-24).
Inventing Admissible Heuristics

- Relaxed problem: a problem with fewer restrictions on the actions
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- If we relax the rules so that a square can move anywhere, we get heuristic $h_1$
- If we relax the rules to allow a square to move to any adjacent square, we get heuristic $h_2$

What you should know

- Be able to run A* by hand on a simple example
- Why it is important for a heuristic to be admissible and consistent
- Pros and cons of A*
- How do come up with heuristics
- What if means for a heuristic function to dominate another heuristic function