Can you have more than one Nash Equilibrium?

• ACME, a video game hardware manufacturer, has to decide whether its next game machine will use DVDs or CDs
• Best, a video game software producer, needs to decide whether to produce its next game on DVD or CD
• Profits for both will be positive if they agree and negative if they disagree
Can you have more than one Nash Equilibrium?

<table>
<thead>
<tr>
<th></th>
<th>Best: dvd</th>
<th>Best: cd</th>
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<tbody>
<tr>
<td>ACME: dvd</td>
<td>A = 9, B = 9</td>
<td>A = -3, B = -1</td>
</tr>
<tr>
<td>ACME: cd</td>
<td>A = -4, B = -1</td>
<td>A = 5, B = 5</td>
</tr>
</tbody>
</table>

There are two Nash Equilibria in this game. In general, you can multiple Nash Equilibria. This creates a big problem. Can you see what that problem is?
Dealing with Multiple Nash Equilibria

1. Could choose the Pareto-optimal Nash Equilibrium eg. (dvd, dvd) but
   - What if there are multiple Pareto-optimal Nash Equilibria?
   - Or it’s too computationally expensive to find all the Nash Equilibria?
   - Or there are an infinite number of Nash Equilibria?

2. Could communicate before the game
   - But what if you can’t compute all the Nash Equilibria beforehand?

3. Take your best guess

This is a big unresolved issue

Can we have no Nash Equilibria?

Two Fingered Morra

Two players, O (for Odd) and E (for Even) simultaneously display one or two fingers. Let the total number of fingers be \( f \).

1. If \( f \) is odd, O collects \( f \) dollars from E.

2. If \( f \) is even, E collects \( f \) dollars from O.

<table>
<thead>
<tr>
<th></th>
<th>O: one</th>
<th>O: two</th>
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<tr>
<td>E: one</td>
<td>E = 2, O = -2</td>
<td>E = -3, O = 3</td>
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<tr>
<td>E: two</td>
<td>E = -3, O = 3</td>
<td>E = 4, O = -4</td>
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E is the max player
Two Fingered Morra

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- No pure strategy Nash Equilibrium
- If total # of fingers is even, O will want to switch
- If total # of fingers is odd, E will want to switch
- Also, this is a zero-sum game (payoffs in a cell sum to zero)

The Big Theorem

- [Nash 1950] In the n-player normal-form game $G=\{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if $n$ is finite and $S_i$ is finite for every $i$ then there exists at least one Nash Equilibrium, possibly involving mixed strategies
Mixed Strategies

- Recall that a pure strategy is a deterministic policy ie. you pick a strategy and play it all the time
- A mixed strategy is a randomized policy ie. you select your strategy based on a probability distribution
- eg. Select strategy S1 with probability p and strategy S2 with probability (1-p)
- Is there a mixed strategy Nash Equilibrium in 2 Fingered Morra?

Formal Definition of a Mixed Strategy

In the normal-form game $G=\{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then a mixed strategy for a player $i$ is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, \ldots, K$ and $p_{i1} + \ldots + p_{iK} = 1$. 
Mixed Strategy Nash Equilibrium

• The pair of mixed strategies (\(M_A, M_B\)) are a Nash Equilibrium iff

• Player A does not want to deviate from \(M_A\) (because \(M_A\) is Player A’s best response to \(M_B\) and)

• Player B does not want to deviate from \(M_B\) (because \(M_B\) is Player B’s best response to \(M_A\))

Finding optimal mixed strategy for two-player zero-sum games

• Note: applies to zero-sum games (or, more generally, constant sum games)

• Von Neumann’s maximin technique
Expected Payoff to E if O Uses a Mixed Strategy

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Suppose O chooses to display one finger with probability p and two fingers with probability (1-p).

If E chooses the pure strategy of one finger, E’s expected profit is
\[ 2p - 3(1-p) = 2p - 3 + 3p = 5p - 3 \]

If E chooses the pure strategy of two fingers, E’s expected profit is
\[ -3p + 4(1-p) = -3p + 4 - 4p = -7p + 4 \]

Suppose O chooses to display one finger with probability p and two fingers with probability (1-p).

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If E chooses the pure strategy of two fingers, E’s expected profit is
\[ -3p + 4(1-p) = -3p + 4 - 4p = -7p + 4 \]

Expected Payoff to E if O Uses a Mixed Strategy

\[ 5p - 3 = -7p + 4 \]
\[ => 12p = 7 \]
\[ => p = 7/12 \]

When \( p < 7/12 \), E plays ‘two’
When \( p > 7/12 \), E plays ‘one’

O gets to pick p to minimize E’s expected payoff. O picks the lowest point of the higher of the two lines. This happens at the intersection of the two lines.

E’s expected payoff at \( p = 7/12 \) is \( 5(7/12) - 3 = -1/12 \)

O’s mixed strategy is \( 7/12 \) for ‘one’, \( 5/12 \) for ‘two’
Expected Payoff to O if E Uses a Mixed Strategy

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Suppose E chooses to display one finger with probability q and two fingers with probability (1-q)

If O chooses the pure strategy of one finger, O’s expected payoff is

\[-2q + 3(1-q) = -2q + 3 - 3q = -5q + 3\]

If O chooses the pure strategy of two fingers, O’s expected payoff is

\[3q - 4(1-q) = 3q - 4 + 4q = 7q - 4\]

\[-5q + 3 = 7q - 4\]
\[\Rightarrow 7 = 12q\]
\[\Rightarrow q = 7/12\]

When q < 7/12, O plays ‘one’
When q > 7/12, O plays ‘two’

E gets to pick p to minimize O’s expected payoff. E picks the lowest point of the higher of the two lines. This happens at the intersection of the two lines.

O’s expected payoff at q=7/12 is \(-5(7/12) + 3 = -35/12 + 36/12 = 1/12\).

E’s mixed strategy is (7/12 for ‘one’, 5/12 for ‘two’).
Mixed Strategy

- E’s expected payoff is -1/12, O’s is 1/12
- It is better to be O than to be E
- The final mixed strategy is for both players to play “one” with probability 7/12 and “two” with probability 5/12
- This is a maximin equilibrium (which is also a Nash equilibrium)

Theoretical Results

- Every two-player zero-sum game has a maximin equilibrium when you allow mixed strategies
- Every Nash equilibrium in a two-player zero-sum game is a maximin equilibrium for both players
Recipe for Computing Optimal Mixed Strategy 2x2 Constant-Sum Games

<table>
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<tr>
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<th>B: S1</th>
<th>B: S2</th>
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<tbody>
<tr>
<td>A: S1</td>
<td>A = m_{11}</td>
<td>A = m_{21}</td>
</tr>
<tr>
<td>A: S2</td>
<td>A = m_{12}</td>
<td>A = m_{22}</td>
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- Let Player B use strategy S1 with probability p
- Compute Player A's expected payo ff if A uses pure strategy S1:
  \[ m_{11}p + m_{21}(1-p) \]
- Compute Player A's expected payo ff if A uses pure strategy S2:
  \[ m_{12}p + m_{22}(1-p) \]
- Find the p between 0 and 1 that minimizes\[
  \max( m_{11}p + m_{21}(1-p), m_{12}p + m_{22}(1-p))
\]
- The optimum will be at p=0, p=1 or at the point where the two lines intersect
- Repeat by letting Player A use Strategy S1 with probability q but looking at B's payoffs now

Recipe for Computing Optimal Mixed Strategy NxM Zero-Sum Games

- NxM game = Player A has N pure strategies, Player B has M pure strategies
- Let Player B use:
  Strategy S1 with probability p_1
  Strategy S2 with probability p_2
  \vdots
  Strategy SN with probability p_N
- Compute Player A's expected payoff if A uses:
  Pure strategy S_1: e_1 = m_{11}p_1 + m_{21}p_2 + \ldots + m_{N1}p_N
  Pure strategy S_2: e_2 = m_{12}p_1 + m_{22}p_2 + \ldots + m_{N2}p_N
  \vdots
  Pure strategy S_M: e_M = m_{1M}p_1 + m_{2M}p_2 + \ldots + m_{NM}p_N
- Find p_1, p_2, \ldots, p_N to minimize\[
  \max( e_1, e_2, \ldots, e_M ) \text{ subject to } \sum p_i = 1 \text{ and } 0 \leq p_i \leq 1 \text{ for all } i
\]
- Use a method called Linear Programming (polynomial time in number of actions)
- Repeat by letting Player A use a mixed strategy and looking at Player B's payoffs
What About Two-Player Non-Zero Sum Games?

- This is a linear complementarity problem
- Use the Lemke-Howson algorithm
- If interested, see “Computing Equilibria for Two-Person Games” by Bernhard von Stengel

Battle of the Sexes

- Traditional game from 1950s (this version is slightly modified and gender neutral)
- A man and a woman are trying to decide on what to do for entertainment. They are at separate workplaces and must decide whether to attend the Oprah Winfrey show or go to a football game
- Both players would rather spend time together than apart
- Player A would rather go to Oprah’s show while Player B would rather go to a football game.

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<th>B: Oprah</th>
<th>B: Football</th>
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<tr>
<td>A: Oprah</td>
<td>$A = 2$, $B = 1$</td>
<td>$A = 0$, $B = 0$</td>
</tr>
<tr>
<td>A: Football</td>
<td>$A = 0$, $B = 0$</td>
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Battle of the Sexes

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There are two pure strategy Nash Equilibria in this game.

Let's calculate the mixed strategy Nash Equilibrium…

Suppose A plays Oprah with probability p and Football with probability (1-p)

- B chooses the pure strategy of Oprah. Expected payoff for B: p
- B chooses the pure strategy of Football. Expected payoff for B: 2(1-p)
- p = 2(1 – p)
  => p = 2 – 2p
  => 3p = 2 => p = 2/3
- Mixed Strategy for A (2/3,1/3), Expected payoff for B is 2/3
**Battle of the Sexes**

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Suppose B plays Oprah with probability $q$ and football with probability $(1-q)$

- A chooses the pure strategy of Oprah. Expected payoff for $A$: $2q$
- A chooses the pure strategy of Football. Expected payoff for $A$: $1-q$

$2q = 1-q$

=> $3q = 1$

=> $q = 1/3$

- Mixed Strategy for B (1/3, 2/3), Expected payoff for $A$ is $2/3$

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**What you should know**

- How to find pure strategy Nash Equilibria in a game
- Problems with having multiple Nash Equilibria
- How to compute mixed strategy Nash Equilibria in two-player constant sum games