Can you have more than one Nash Equilibrium?

- ACME, a video game hardware manufacturer, has to decide whether its next game machine will use DVDs or CDs
- Best, a video game software producer, needs to decide whether to produce its next game on DVD or CD
- Profits for both will be positive if they agree and negative if they disagree

<table>
<thead>
<tr>
<th>Best: dvd</th>
<th>Best: cd</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACME: dvd</td>
<td>A = 9, B = 9</td>
</tr>
<tr>
<td>ACME: cd</td>
<td>A = -4, B = -1</td>
</tr>
</tbody>
</table>

There are two Nash Equilibria in this game. In general, you can multiple Nash Equilibria. This creates a big problem. Can you see what that problem is?

Dealing with Multiple Nash Equilibria

1. Could choose the Pareto-optimal Nash Equilibrium eg. (dvd, dvd) but
   - What if there are multiple Pareto-optimal Nash Equilibria?
   - Or it’s too computationally expensive to find all the Nash Equilibria?
   - Or there are an infinite number of Nash Equilibria?
2. Could communicate before the game
   - But what if you can’t compute all the Nash Equilibria beforehand?
3. Take your best guess

   This is a big unresolved issue

Can we have no Nash Equilibria?

**Two Fingered Morra**

Two players, O (for Odd) and E (for Even) simultaneously display one or two fingers. Let the total number of fingers be \( f \).

1. If \( f \) is odd, O collects \( f \) dollars from E.
2. If \( f \) is even, E collects \( f \) dollars from O.

<table>
<thead>
<tr>
<th>E: one</th>
<th>O: two</th>
</tr>
</thead>
<tbody>
<tr>
<td>E = 2, O = -2</td>
<td>E = -3, O = 3</td>
</tr>
<tr>
<td>E = -3, O = 3</td>
<td>E = 4, O = -4</td>
</tr>
</tbody>
</table>
Two Fingered Morra

<table>
<thead>
<tr>
<th></th>
<th>O: one</th>
<th>O: two</th>
</tr>
</thead>
<tbody>
<tr>
<td>E: one</td>
<td>E = 2, O = -2</td>
<td>E = -3, O = 3</td>
</tr>
<tr>
<td>E: two</td>
<td>E = -3, O = 3</td>
<td>E = 4, O = -4</td>
</tr>
</tbody>
</table>

- No pure strategy Nash Equilibrium
- If total # of fingers is even, O will want to switch
- If total # of fingers is odd, E will want to switch
- Also, this is a zero-sum game (payoffs in a cell sum to zero)

The Big Theorem

- [Nash 1950] In the n-player normal-form game \( G = \{ S_1, \ldots, S_n; u_1, \ldots, u_n \} \), if \( n \) is finite and \( S_i \) is finite for every \( i \) then there exists at least one Nash Equilibrium, possibly involving mixed strategies

Mixed Strategies

- Recall that a pure strategy is a deterministic policy ie. you pick a strategy and play it all the time
- A mixed strategy is a randomized policy ie. you select your strategy based on a probability distribution
- eg. Select strategy \( S_1 \) with probability \( p \) and strategy \( S_2 \) with probability \( (1-p) \)
- Is there a mixed strategy Nash Equilibrium in 2 Fingered Morra?

Formal Definition of a Mixed Strategy

In the normal-form game \( G = \{ S_1, \ldots, S_n; u_1, \ldots, u_n \} \), suppose \( S_i = \{ s_{i1}, \ldots, s_{ik} \} \). Then a mixed strategy for a player \( i \) is a probability distribution \( p_i = (p_{i1}, \ldots, p_{ik}) \), where \( 0 \leq p_{ik} \leq 1 \) for \( k = 1, \ldots, K \) and \( p_{i1} + \ldots + p_{ik} = 1 \).

Mixed Strategy Nash Equilibrium

- The pair of mixed strategies \( (M_A, M_B) \) are a Nash Equilibrium iff
- Player A does not want to deviate from \( M_A \) (because \( M_A \) is Player A’s best response to \( M_B \) and)
- Player B does not want to deviate from \( M_B \) (because \( M_B \) is Player B’s best response to \( M_A \))

Finding optimal mixed strategy for two-player zero-sum games

- Note: applies to zero-sum games (or, more generally, constant sum games)
- Von Neumann’s maximin technique
Expected Payoff to E if O Uses a Mixed Strategy

<table>
<thead>
<tr>
<th>O: one</th>
<th>O: two</th>
</tr>
</thead>
<tbody>
<tr>
<td>E: one</td>
<td>E = 2, O = -2&lt;br&gt; E = -3, O = 3</td>
</tr>
<tr>
<td>E: two</td>
<td>E = -3, O = 3&lt;br&gt; E = 4, O = -4</td>
</tr>
</tbody>
</table>

Suppose O chooses to display one finger with probability $p$ and two fingers with probability $(1-p)$.

If E chooses the pure strategy of one finger, E’s expected profit is $2p - 3(1-p) = 2p - 3 + 3p = 5p - 3$.

If E chooses the pure strategy of two fingers, E’s expected profit is $-3p + 4(1-p) = -3p + 4 - 4p = -7p + 4$.

Expected Payoff to O if E Uses a Mixed Strategy

<table>
<thead>
<tr>
<th>O: one</th>
<th>O: two</th>
</tr>
</thead>
<tbody>
<tr>
<td>E: one</td>
<td>E = 2, O = -2&lt;br&gt; E = -3, O = 3</td>
</tr>
<tr>
<td>E: two</td>
<td>E = -3, O = 3&lt;br&gt; E = 4, O = -4</td>
</tr>
</tbody>
</table>

Suppose E chooses to display one finger with probability $q$ and two fingers with probability $(1-q)$.

If O chooses the pure strategy of one finger, O’s expected payoff is $-2q + 3(1-q) = -2q + 3 - 3q = -5q + 3$.

If O chooses the pure strategy of two fingers, O’s expected payoff is $3q - 4(1-q) = 3q - 4 + 4q = 7q - 4$.

Mixed Strategy

- E’s expected payoff is $-1/12$, O’s is $1/12$.
- It is better to be O than to be E.
- The final mixed strategy is for both players to play “one” with probability $7/12$ and “two” with probability $5/12$.
- This is a maximin equilibrium (which is also a Nash equilibrium).

Theoretical Results

- Every two-player zero-sum game has a maximin equilibrium when you allow mixed strategies.
- Every Nash equilibrium in a two-player zero-sum game is a maximin equilibrium for both players.
Recipe for Computing Optimal Mixed Strategy 2x2 Constant-Sum Games

- Let Player B use strategy S1 with probability p
- Compute Player A's expected payoff if A uses pure strategy S1:
  \( m_{11}p + m_{21}(1-p) \)
- Compute Player A's expected payoff if A uses pure strategy S2:
  \( m_{12}p + m_{22}(1-p) \)
- Find the p between 0 and 1 that minimizes
  \( \max( m_{11}p + m_{21}(1-p), m_{12}p + m_{22}(1-p) ) \)
- The optimum will be at p=0, p=1 or at the point where the two lines intersect
- Repeat by letting Player A use Strategy S1 with probability q but looking at B’s payoffs now

Recipe for Computing Optimal Mixed Strategy NxM Zero-Sum Games

- NxM game = Player A has N pure strategies, Player B has M pure strategies
- Let Player B use:
  - Strategy S1 with probability \( p_1 \)
  - Strategy S2 with probability \( p_2 \)
  - Strategy SN with probability \( p_N \)
- Compute Player A's expected payoff if A uses:
  - Pure strategy S1: \( e_1 = m_{11}p_1 + m_{21}p_2 + \ldots + m_{N1}p_N \)
  - Pure strategy S2: \( e_2 = m_{12}p_1 + m_{22}p_2 + \ldots + m_{N2}p_N \)
  - Pure strategy SM: \( e_M = m_{1M}p_1 + m_{2M}p_2 + \ldots + m_{NM}p_N \)
- Find \( p_1, p_2, \ldots, p_N \) to minimizes
  \( \max( e_1, e_2, \ldots, e_M ) \) subject to
    \( \Sigma p_i = 1 \) and \( 0 \leq p_i \leq 1 \) for all i
- Use a method called Linear Programming (polynomial time in number of actions)
- Repeat by letting Player A use a mixed strategy and looking at Player B's payoffs

What About Two-Player Non-Zero Sum Games?

- This is a linear complementarity problem
- Use the Lemke-Howson algorithm
- If interested, see “Computing Equilibria for Two-Person Games” by Bernhard von Stengel

Battle of the Sexes

- Traditional game from 1950s (this version is slightly modified and gender neutral)
- A man and a woman are trying to decide on what to do for entertainment. They are at separate workplaces and must decide whether to attend the Oprah Winfrey show or go to a football game
- Both players would rather spend time together than apart
- Player A would rather go to Oprah’s show while Player B would rather go to a football game.

There are two pure strategy Nash Equilibria in this game.
Let’s calculate the mixed strategy Nash Equilibrium...

Suppose A plays Oprah with probability p and Football with probability (1-p)
- B chooses the pure strategy of Oprah. Expected payoff for B: \( p \)
- B chooses the pure strategy of Football. Expected payoff for B: \( 2(1-p) \)
- \( p = 2(1-p) \)
  \[ p = 2 - 2p \]
  \[ 3p = 2 \] \( \Rightarrow p = 2/3 \)
- Mixed Strategy for A (2/3,1/3), Expected payoff for B is 2/3
Battle of the Sexes

<table>
<thead>
<tr>
<th></th>
<th>B: Oprah</th>
<th>B: Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: Oprah</td>
<td>A = 2, B = 1</td>
<td>A = 0, B = 0</td>
</tr>
<tr>
<td>A: Football</td>
<td>A = 0, B = 0</td>
<td>A = 1, B = 2</td>
</tr>
</tbody>
</table>

Suppose B plays Oprah with probability q and football with probability (1 - q)

- A chooses the pure strategy of Oprah. Expected payoff for A: 2q
- A chooses the pure strategy of Football. Expected payoff for A: 1-q
- 2q = 1-q
  \[
  \Rightarrow 3q = 1 \\
  \Rightarrow q = 1/3
  \]
- Mixed Strategy for B (1/3, 2/3), Expected payoff for A is 2/3

What you should know

- How to find pure strategy Nash Equilibria in a game
- Problems with having multiple Nash Equilibria
- How to compute mixed strategy Nash Equilibria in two-player constant sum games