1. Suppose that $X$, the inter arrival time between two packets from two different sources at a router, satisfies

$$P(x > t) = \alpha e^{-t} + \beta e^{-2t}, t \geq 0$$

Where $\alpha + \beta = 1$, $\beta \geq 0$. Calculate the mean of $X$.

**Solution:**

**Hint:** In order to calculate $E[X]$, we use $m_1 = E\{X\} = \int_{-\infty}^{\infty} x f_X(u) du$ → we need to find $f_X(u) \rightarrow$ find $F_X(x)$.

a) Find $F_X(x)$.

$$F_X(t) = 1 - P(x > t) = 1 - \alpha e^{-t} - \beta e^{-2t}, t \geq 0$$

b) Find $f_X(x)$.

$$f_X(t) = \frac{dF_X(t)}{dt} = \alpha e^{-t} + 2\beta e^{-2t}, t \geq 0$$

c) Calculate $E[X]$.

We know the Exponential distribution: $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$ and the mean value is

$$E\{X\} = \frac{1}{\lambda}.$$ 

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(t) dt = \int_{0}^{\infty} x \alpha e^{-t} + 2\beta e^{-2t} dt = \int_{0}^{\infty} \alpha te^{-t} dt + \int_{0}^{\infty} 2\beta e^{-2t} dt = \alpha \frac{1}{2} + \frac{\beta}{2}$$

2. Suppose the current $I$ (A) and the resistor $R$ (Ohm) are independent from each other in a circuit.

The PDF of current $I$ is $g_I(i)$ and the PDF of resistor $R$ is $h_R(r)$. What is the mean value of voltage $V$.

$$g_I(i) = \begin{cases} 2i, & 0 \leq i \leq 1, \\ 0, & \text{o.w.} \end{cases}$$

$$h_R(r) = \begin{cases} \frac{r^2}{9}, & 0 \leq r \leq 3, \\ 0, & \text{o.w.} \end{cases}$$

**Solution:**

**Hint:** We know $V = IR$, hence $E\{V\} = E\{IR\} = E\{I\} \cdot E\{R\}$ (current and resistor
are independent) → We need to calculate $E[I]$ and $E[R]$.

a) Calculate $E[I]$ and $E[R]$.

$$E\{I\} = \int_{-\infty}^{\infty} ig_1(i) \, di = \int_{0}^{1} 2i^2 \, di = \frac{2}{3} A$$

$$E\{R\} = \int_{-\infty}^{\infty} rh_2(r) \, dr = \int_{0}^{\frac{1}{2}} \frac{r^3}{9} \, dr = \frac{9}{4} \Omega$$

b) $E\{V\} = E\{IR\} = E\{I\} \cdot E\{R\} = \frac{2}{3} \cdot \frac{9}{4} = \frac{3}{2} V$

3. Consider a game in which you roll a pair of dice. If the sum of two numbers is greater than 20, then you win $5. If the sum of two numbers is less than 7, then you win $4. Otherwise, you lose $5.

(1) What are your expected winnings?
(2) If the sum of two numbers is 12, then you win $10. If the sum of two numbers is less than 4, then you lose $4. Otherwise, you win $1. Up to how much should you be willing to play each game?

**Solution:**

**Hint:** In order to calculate expected winnings, we use $E\{X\} = x_1p_1 + x_2p_2 + x_3p_3$. In this case, $x_1=5, x_2=4, x_3=-5$. Hence, we need to calculate $p_1, p_2, p_3$.

1) a) Calculate $p_1, p_2, p_3$.

The PMF is shown below.

![PMF](image)

$$p_1 = P(\text{the sum of two numbers is greater than 10}) = \frac{3}{36} = \frac{1}{12}$$

$$p_2 = P(\text{the sum of two numbers is less than 7}) = \frac{15}{36} = \frac{5}{12}$$

$$p_3 = P(\text{others}) = 1 - p_1 - p_2 = 1 - \frac{1}{12} - \frac{5}{12} = \frac{1}{2}$$

b) $E\{X\}$.

$X$: Your expected winnings.

$$E\{X\} = x_1p_1 + x_2p_2 + x_3p_3 = 5 \cdot \frac{1}{12} + 4 \cdot \frac{5}{12} - 5 \cdot \frac{1}{2} = -0.42$$
2)  
   a) Find the PMF.

   b) E[X].

   \[ E\{X\} = x_1p_1 + x_2p_2 + x_3p_3 = \frac{10}{36} - 4\times \frac{3}{36} + \frac{32}{36} = \frac{30}{36} = $0.83 \]

4. A particular model of an HDTV is manufactured in three different plants, say, A, B, and C, of the same company. Because the workers at A, B, and C are not equally experienced, the quality of the units differs from plant to plant. The pdf’s of the time-to-failure \( X_A, X_B, X_C \), in years, are

   \[
   f_{X_A}(x) = \begin{cases} 
   \frac{1}{5} \exp\left(-\frac{x}{5}\right), & x > 0, \text{ for A} \\
   0, & o, w.
   \end{cases}
   \]

   \[
   f_{X_B}(x) = \begin{cases} 
   \frac{1}{6.5} \exp\left(-\frac{x}{6.5}\right), & x > 0, \text{ for B} \\
   0, & o, w.
   \end{cases}
   \]

   \[
   f_{X_C}(x) = \begin{cases} 
   \frac{1}{10} \exp\left(-\frac{x}{10}\right), & x > 0, \text{ for C} \\
   0, & o, w.
   \end{cases}
   \]

   Plant A produces three times as many units as B. Plant B produces twice as many as C. The TVs are all sent to a central warehouse, intermingled, and shipped to retail stores all around the country.

   What is the expected lifetime of a unit purchased at random?

   **Solution:**

   **Hint:** In order to calculate \( E[X] \), we use the concept called conditional expectation

   \[
   E\{X\} = E\{X_A\} P(A) + E\{X_B\} P(B) + E\{X_C\} P(C), \quad \text{also be written as} \]

   \[
   E\{X\} = E\{X|A\} P(A) + E\{X|B\} P(B) + E\{X|C\} P(C). \quad \rightarrow \text{We need to find} \]

   \( E[X_A], E[X_B], E[X_C] \) and \( P(A), P(B), P(C) \).

   a) Let the number of units manufactured at the various sites be denoted \( n_A, n_B \) and \( n_C \), with total number of units simply \( n \). Then from the problem statement we know that

   \[
   n_A = 3n_B \quad \text{and} \quad n_B = 2n_C, \]

   \[
   n = n_A + n_B + n_C.
   \]
and of course \( n = n_A + n_B + n_C \). We define event A is the units comes from plant A, and so forth for event B and C.

\[
P(A) = \frac{n_A}{n} = \frac{6}{9}, \quad P(B) = \frac{n_B}{n} = \frac{2}{9}, \quad P(C) = \frac{n_C}{n} = \frac{1}{9}
\]

\[
E\{X_A\} = \int_0^\infty x \exp\left(-\frac{x}{5}\right) dx = 5
\]

\[
E\{X_B\} = \int_0^\infty x \exp\left(-\frac{x}{6.5}\right) dx = 6.5
\]

\[
E\{X_C\} = \int_0^\infty x \exp\left(-\frac{x}{10}\right) dx = 10
\]

b) \[
E\{X\} = E\{X_A\} P(A) + E\{X_B\} P(B) + E\{X_C\} P(C)
\]

\[
= 5 \cdot \frac{6}{9} + 6.5 \cdot \frac{2}{9} + 10 \cdot \frac{1}{9} = 5.89 \text{ years.}
\]