THE FIRST-ORDER DELTA-SIGMA MODULATOR

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Outline

- Quantizers and quantization noise
- Binary quantization
- MOD1 as an ADC
- MOD1 as a DAC
- MOD1 linear model
- Simulation of MOD1
- MOD1 under DC excitation
- The effects of finite op-amp gain
- Decimation filters for MOD1
Quantizers and Quantization Noise (1)

- Unipolar N-bit quantizer:

Quantizer

\[ y \rightarrow v \]

Digital Output

\[ 2^{N-1} \]

Analog Input

Quantizers and Quantization Noise (2)

- M-step mid-rise quantizer:

\[ e = y - y \]

no-overload range

- M-step mid-tread quantizer:

\[ v = k_y + e \]
Quantizers and Quantization Noise (3)

- Sampled signal:

- Quantization error: irrational!

Quantizers and Quantization Noise (4)

- FFT:

- Sampled signal (f = f_s/8):
Binary Quantization (1)

- Quantization error:

- FFT:

Binary Quantization (2)

- Modeling the gain:

- Minimize mean square error of $e$:
  $$\sigma^2_e = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} e(n)^2$$

  $$\langle a, b \rangle \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} a(n)b(n) = E\{ab\}$$

  $$\begin{align*}
  \sigma^2_e &= \langle e, e \rangle \\
  &= \langle v - ky, v - ky \rangle \\
  &= \langle v, v \rangle - 2k \langle v, y \rangle + k^2 \langle y, y \rangle.
  \end{align*}$$

  opt.: $k = \frac{\langle v, y \rangle}{\langle y, y \rangle} = \frac{E[v]}{E[y^2]}$
MOD1 as an ADC (1)

- Linear modeling:

\[ u \xrightarrow{\int} y \]

\[ U(z) \xrightarrow{1/z-1} V(z) \]

MOD1 as an ADC (2)

- Continuous-time implementation:

- Discrete-time switched-capacitor implementation:
MOD1 as an ADC (3)

- Continuous-time waveforms:

- Z-domain model:

MOD1 as an ADC (4)

- Stable operation:

\[ v(n) = \text{sgn}[y(n)]. \]

\[ y(n) = y(n-1) + u(n) - v(n-1) \]

\[ y(N) - y(0) = \sum_{n=0}^{N} [u(n) - v(u-1)]. \]

If \( y(n) \) is bounded,

\[ \lim_{N \to \infty} \frac{y(N) - y(0)}{N} = 0 \]

\[ \bar{u} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} u(n) = \bar{v} \]

Perfectly accurate for \( N \to \infty. \)
MOD1 as a DAC

- Error feedback structure: \( \rightarrow \) recycled error!

\[
y(n) = u(n) + y_{LSB}(n-1)
\]

\[
y(n) = u(n) + y(n-1) - v(n-1)
\]

Same as for \( \Delta \Sigma \) loop \( \rightarrow \) another option for DAC.
(For ADC, impractical!)

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MOD1 Linear Model (1)

- Z-domain analysis:

\[
Y(z) = z^{-1}Y(z) + U(z) - z^{-1}V(z)
\]

\[
V(z) = Y(z) + E(z) = z^{-1}Y(z) + U(z) - z^{-1}V(z) + E(z)
\]

\[
= U(z) + E(z) - z^{-1}(V(z) - Y(z))
\]

\[
= U(z) + E(z) - z^{-1}E(z)
\]

\[
= U(z) + (1 - z^{-1})E(z).
\]

\[
V(z) = STF(z)U(z) + NTF(z)E(z)
\]
**MOD1 Linear Model (2)**

- Frequency-domain analysis:

\[
|NTF(e^{j2\pi f})|^2 = [2 \sin(\pi f)]^2
\]

Mean square of \(q_o^2\):

\[
\sigma_{q_o}^2 = \int_0^{1/(2 \cdot OSR)} [2\pi f]^2 S_c(f) df = \frac{\pi^2}{9(\text{OSR})^3} \text{ (for OSR >> 1)}
\]

\(SQRN = \frac{\sigma_{q_o}^2}{\sigma_{\epsilon_0}^2} = \frac{9M^2(\text{OSR})^3}{2\pi^2}\) Signal-to-quantization noise ratio

**Simulation of MOD1 (1)**

- Output spectrum for full-scale sine-wave input:

Looks ok, but SQNR 5 dB less than the formulaic.
Simulation of MOD1 (2)

- SQNRs for different frequencies:

Simulation of MOD1 (3)

- In-band quantization noise power:

\[ V_{\text{ref}} = \pm 1 \]
MOD1 Under DC Excitation (1)

- Idle tones:
  \[ y(n) = y(n-1) + u - y(n-1) \]
  \[ v(n) = \text{sgn}(y(n)) \]
  \[ y(n) = y(n-1) + u - \text{sgn}(y(n-1)) \]

- \( u = y(0) = \frac{1}{2} \):

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<th>1</th>
<th>2</th>
<th>3</th>
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<td>( \frac{1}{2} )</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( v(n) )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- For \( u = 0.01 \), tones at \( k f_s/200 \), \( k = 1, 2, \ldots \)

MOD1 Under DC Excitation (2)

- Let \( u = a/b \), \( a \) and \( b \) odd integers, and \( 0 < a < b \). Also, let \( |y(0)| < 1 \). Then, the output has a period \( b \) samples. In each period, \( v(n) \) will contain \((b+a)/2\) samples of +1, and \((b-a)/2\) samples of -1.

- If \( a \) or \( b \) is even, the period is \( 2b \), with \((a+b)\) +1s and \((b-a)\) -1s.

- If \( v(n) \) has a period \( p \), with \( n+1 \)s and \( p-m \) -1s, the average \( \bar{v} = (2m - p)/p \). Hence, \( u = \bar{v} \) is also rational. Thus, rational dc \( u \iff \) periodic \( v(n) \).

- Periodic \( v(n) \): pattern noise, idle tone, limit cycle. Not instability!

- For \( u = 1/100 \), tones at \( k f_s/200 \), \( k = 1, 2, \ldots \) some may be in the baseband. Often intolerable!
Stability of MOD1

- MOD1 always stable as long as $|u| \leq 1$, and $|y(0)| \leq 2$:

$$y(n) = \left[y(n-1) - \text{sgn}(y(n-1))\right] + u(n) \quad \leq 2$$

- If $u > 1$ (or $u < 1$), $v$ will always be $+1$ (or $-1$) $\Rightarrow y$ will increase (or decrease) indefinitely.

- If $|u(n)| \leq 1$ but $|y(0)| > 2$, then $|y(n)|$ will decrease to $< 2$. Output spectrum always a line spectrum for MOD1 with dc input (rational or not).

The Effects of Finite Op-Amp Gain (1)

- Degraded noise shaping:

$$q_2(n) = q_2(n-1) + C_1\left[u(n) - v(n-1) - \frac{q_2(n)}{C_2(A+1)}\right]$$

$$Y(z) = \frac{p^2 U(z) - V(z)}{z - p}$$

$$NTF(z) = 1 - p z^{-1} \rightarrow 1 - p = 1/A$$

Pole error, dc gain of NTF
The Effects of Finite Op-Amp Gain (2)

- Dead zones:
  Ideally: \( y(1) = y(0) + u - \text{sgn}(y(0)) = u - 1 < 0 \) \( v = -1 \)
  \( y(2) = u - 1 + u + 1 = 2u > 0 \) \( +1 \)
  \( y(3) = 2u + u - 1 = 3u - 1 < 0 \) \( -1 \)
  \( y(k) = \begin{cases} ku - 1, & \text{if } k \text{ is odd} \\ ku, & \text{if } k \text{ is even} \end{cases} \)
  for \( u > 0 \), eventually \( ku > 1 \) and two 1’s occur.

For \( A \approx 10^3 \):
  \( y(1) = y(0) + u - \text{sgn}(y(0)) = u - 1 < 0 \) \( v = -1 \)
  \( y(2) = pu - p + u + 1 = (1 + p)u + (1 - p) > 0 \) \( +1 \)
  \( y(3) = p(u + pu + p(1 - p) + u - 1 = (1 + p^2)u - (1 - p + p^2) < 0 \) \( -1 \)
  \[ y(k) = \sum_{i=0}^{k-1} p^i u + (-1)^k \sum_{i=0}^{k-1} (-p)^i \]

The Effects of Finite Op-Amp Gain (3)

- For \( v > 0 \), \( (\text{Two 1’s occurring}) \)
  \( \frac{u}{1-p} > \frac{1}{1 + p} \)
  \( u > \frac{1 - p}{1 + p} = \frac{1/A}{2 - 1/A} = \frac{1}{2A} \)

For \( A = 10^3 \):
  \( u_{min} \sim 1/(2A) \)
Decimation Filters for MOD1 (1)

- The sinc filter:
  
  \[ w(n) = \frac{1}{N} \sum_{i=0}^{N-1} v(n-i) \]
  
  \[ h_1(n) = \begin{cases} 
  1/N, & \text{if } 0 \leq n \leq N-1 \\
  0, & \text{otherwise} 
  \end{cases} \]
  
  \[ H_1(z) = \frac{1 - z^{-N}}{N(1 - z^{-N})} \]
  
  \[ H_1(e^{j2\pi f}) = \frac{\text{sinc}(Nf)}{\text{sinc}(f)} \]

Decimation Filters for MOD1 (2)

- Responses:
  
  \[ h_1(n) \]
  
  Gain response \( H_1(z) \)
  
  Areas around notches fold back to baseband after decimation if \( N = \text{OSR} \).
Decimation Filters for MOD1 (3)

- Implementation:

\[ Q_1(z) = H_1(z)NTF(z)E(z) = \frac{1}{N}(1-z^{-N})E(z) \]

\[ q_1(n) = \frac{1}{N}[e(n) - e(n-N)] \]

Assuming \( e(n) \) and \( e(n-N) \) are uncorrelated:

\[
\sigma_{q_1}^2 = \frac{2\sigma_c^2 e_{\text{rms}}^2}{N^2} \quad \sigma_{q_0}^2 = \frac{\pi^2\sigma_c^2}{3N^3} \]

Inband noise after \( H_1 \): \( \sigma_{q_1}^2 \)

Inband noise before \( H_1 \): \( \sigma_{q_0}^2 \)

Total noise after \( H_1 \): Too much!

Total noise after ideal LPF: Much less than \( \sigma_{q_0}^2 \)

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Decimation Filters for MOD1 (4)

- The sinc^2 filter:

\[ H_2(z) = \frac{(1-z^{-N})^2}{N(1-z^{-1})} \]

\[ H_2(e^{j2\pi f}) = \left( \frac{\text{sinc}(Nf)}{\text{sinc}(f)} \right)^2 \]

\[ Q_2(z) = NTF(z)H_2(z)E(z) = \frac{1}{N^2}(1-z^{-N})(1-z^{-N})E(z) = \frac{1}{N}H_1(z)[(1-z^{-N})E(z)] \]

\[ q_2(n) = \frac{1}{N^2} \sum_{i=0}^{N-1} [e(n-i) - e(n-N-i)] \]

\[ \sigma_{q_2}^2 = \frac{2N\sigma_c^2 e_{\text{rms}}^2}{N^4} = \frac{2\sigma_c^2}{N^3} \]

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Decimation Filters for MOD1 (5)

• Response:

![Graph showing frequency response over sample number]

• Implementation:

![Diagram of decimation filter circuit]

References