1. An ID number includes 9 digits selected uniformly random in range from 0-9. Each number can be selected multiple times. What is the probability that none of the digit can be 0 or 3 if the first digit is not 0?

Define two events firstly:

$A =$ none of the digit can be 0 or 3
$B =$ the first digit is not 0

Then, the probability that none of the digit can be 0 or 3 if the first digit is not 0 can be calculated as:

$$P(A | B) = \frac{P(AB)}{P(B)}$$

2. A standard 52-card deck includes thirteen ranks, A-K-Q-J-10-9-8-7-6-5-4-3-2, of each of the four suits: clubs (♣), diamonds (♦), hearts (♥) and spades (♠). Randomly select five cards without replacement. What is probability for selecting:

(a) Straight Flush? (Five cards of the same suit in sequence. For example: ♠J-♦10-♥9-♣8-♥7. In this game, 5-4-3-2-A is not allowed)

For counting the number of samples in this event, we firstly consider how much combinations we have within one suit as:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$$

Due to four different suits, multiplying the number by four gives the number of samples in this event. And then, we can calculate the probability as:

$$P = \frac{N(A)}{N(\Omega)} = \frac{4 \times 13 - 5 + 1}{52 \choose 5}$$

where we call selecting Straight Flush as event A and $\Omega$ is the whole sample space.

(b) Full House? (Three cards of one rank and two cards of another rank. For example: ♦7-♦7-♥7-♥9-♣9)

For counting the number of samples in this part, we can consider the process as:

Step 1: choose one rank from thirteen.
Step 2: assign three suits from four for these three cards.
Step 3: choose one rank from the rest twelve ranks.
Step 4: assign two suits from four for these two cards.

(c) Flush? (Five cards of the same suit. For example: ♣K-♥J-♦9-♠3-♣2. Flush is not including Straight Flush)
3. Refer to the first problem in HW1.

(a) In a general case, for \( n \) distinguishable particles and \( r \) possible cells, there are how many possible outcomes? Assume \( r \geq n \).

Compare to the first question. For one ID, you need to decide the number for 9 digits from 10 possible numbers. In this question, we are now distributing \( n \) distinguishable particles (digits) into \( r \) possible cells. For each particle, you have \( r \) choices. For two particles, you will then have \( r \times r = r^2 \) choices. Extend to \( n \) particles, the number of all possible outcomes will be

\[
N(\Omega) = r \times \ldots \times r = r^n
\]

This can be verified by the size of sample space for the first question which equals to \( 10^9 \).

(b) What is the probability for event \( A, B \) and \( C \) as we defined in HW1.

Recall the events we defined in the HW1:

- \( A \) = multiple particles occupy a cell
- \( B \) = the last cell is not empty
- \( C \) = \( A \) occurs but not \( B \).

For counting the number of samples in event \( A \), we count the number of samples in event \( A^c \) firstly, and then, using the relationship \( N(A) = N(\Omega) - N(A^c) \) to get the number of samples in event \( A \).

Event \( A^c \) can be described by: each cell can be occupied by at most one particle. Using the idea of permutation, we have

\[
N(A^c) = P_n^r = \frac{r!}{(r-n)!}
\]

(order is matter due to distinguishable particles)

You can verify this equation by plug in the number given in the HW1:

\[
N(A^c) = P_3^3 = \frac{3!}{(3-3)!} = 6 \rightarrow N(A) = 27 - 6 = 21
\]

For counting the number of samples in event \( B \), we use the same methodology:

Event \( B^c \) can be described by: the last cell is empty which means you have only \( r-1 \) choices for each particle. So, we have \( N(B^c) = (r-1)^n \). And then, \( N(B) = N(\Omega) - N(B^c) \)

Verify this equation using the number given in HW1:

\[
N(B^c) = (3-1)^3 = 8 \rightarrow N(B) = 27 - 8 = 19
\]

For event \( C \), you will need to use the idea of total probability mentioned on text book page 29-31:

\[
P(B^c) = P(A \mid B^c) + P(A^c \mid B^c) \rightarrow P(C) = P(B^c) - P(A^c \mid B^c)
\]

Also, think about the case when \( r = n \).
4. Refer to the fourth problem in HW2. Consider a large communication channel which is composed by $N$ identical independent binary channel sections as shown in below:

Each binary channel has the same channel transition probabilities $p_0$, $p_1$, $q_0$ and $q_1$ as defined in HW2.

Assume $p_0 = p_1$ and $q_0 = q_1$

(a) Assuming 0 and 1 at input of the 1st channel section occurs with same probability 0.5, what is the error rate at the output of the 4th channel section, $P(X_1 \neq Y_4)$?

Take $N=2$ for example:

We can simply list the cases for error happens:

and non-error happens:
You can see when both binary channels occur error, the channel has the ability to correct these errors and finally we have a successful transmission. (last two cases.)

This means the error for the whole channel occurs when odd number of binary channels have error transmission. We can model this to a N-time Bernoulli trials. The error rate for whole communication channel will be the probability of odd number fails (errors) happen.

For \( N=4 \), we can calculate the error rate as:

\[
P_1 = \binom{4}{1} \left( \frac{1}{2} (p_0q_1^3 + q_1^3) \right) + \binom{4}{3} \left( \frac{1}{2} (q_0p_1^3 + q_1p_1^3) \right)
\]

The first term counts the condition that one of the channel section occurs error, and the second term counts the condition that three channel sections occur error.

(b) Under the same assumption, what is the error rate for the whole communication channel, \( P(X_1 \neq Y_N) \)?

Try to extend the solution to a general case for \( N \) sections. Think about the cases when \( N \) is an even number or odd number.