1. In an automatic meter reading system, the electrical meter measures and transmits the message to the receiver. If the transmission fails, the receiver requires a new transmission until a successful transmission. Meter can repeat at most 5 times. Each transmission has a success probability \( p = 0.75 \). If \( X \) is the random variable of the number of transmission, what is the probability mass function (PMF) for \( X \)? CDF?

\[
PMF = f(X) = \begin{cases} 
(1-p)^{X-1} p & 1 \leq X \leq 4 \\
(1-p)^4 & X = 5 \\
0 & \text{o.w.}
\end{cases}
\]

\[
CDF = F(X) = \sum_{i=1}^{X} (1-p)^{i-1} p \quad 1 \leq X \leq 4
\]

\[
1 \quad 5 \leq X
\]
2. Let the number of packets received by a server follow a Poisson distribution with \( \lambda = 10 \) packets per \( 1 \mu s \).

(a) Calculate the probability of receiving more than 20 in \( 1 \mu s \).

\[
P(X > 20) = \sum_{k=21}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=21}^{20} \frac{10^k e^{-10}}{k!} = 1 - \sum_{k=0}^{20} \frac{10^k e^{-10}}{k!}
\]

(b) Calculate the probability of receiving more than \( 20 \times 10^6 \) packets in \( 1 s \).

\[
P(X > 20 \times 10^6) = \sum_{k=20 \times 10^6 + 1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=20 \times 10^6 + 1}^{\infty} \frac{(N \lambda_{1 \mu s})^k e^{-N \lambda_{1 \mu s}}}{k!}
\]

(c) Assuming that a program handle all the packets received in a queue every \( 1 \mu s \), how big should the queue be to guarantee that all packets are handled with probability 0.9.

Handle all the packets received in \( 1 \mu s \) means the size of queue should be larger than the number of packets received in \( 1 \mu s \). So, the probability of handle all the packets received in \( 1 \mu s \) is equivalent to the probability of the number of received packets in \( 1 \mu s \) is smaller than the size of queue, \( N \):

\[
P(X \leq N) \geq 0.9
\]

Then, based on the CDF of Poisson distribution shown below, we design our queue with size \( N = 15 \).
3. The lifetime of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than $1.8 \times 10^6$ hours?

For a single chip, we have

$$P(T_{\text{life}} < 1.8 \times 10^6) = \Phi\left(\frac{1.8 \times 10^6 - 1.4 \times 10^6}{3 \times 10^5}\right) = \Phi\left(\frac{4}{3}\right)$$

Then, we can think this is a Binomial random variable with $n = 1000$ and $p = \Phi\left(\frac{4}{3}\right)$

Then, $P(i \geq 20) = \sum_{i=20}^{100} \binom{100}{i} p^i (1 - p)^{100-i} = 1 - \sum_{i=1}^{19} \binom{100}{i} p^i (1 - p)^{100-i}$
4. In a communication system, bipolar signal \( X \) are sent with equal probability from transmitter: \( P(X = -1) = P(X = 1) = 0.5 \). Through a wireless channel, at receiver, we received a distorted signal \( X' \).

The conditional PDF for received signal \( X' \) can be written as:

\[
\begin{align*}
P(X' | X = 1) &\sim N(\alpha, \sigma^2) \\
P(X' | X = -1) &\sim N(-\alpha, \sigma^2)
\end{align*}
\]

where \( 0 < \alpha < 1 \).

We set a threshold \( TH \) based on which we can decide what signal is transmitted:

\[
Y = \begin{cases} 
1 & X' > TH \\
-1 & X' < TH
\end{cases}
\]

where \( Y \) is the output of our receiver.

Calculate the error rate \( P(Y \neq X) \) for this communication channel. Make a guess for the optimal threshold which minimum the error rate.

Optimal threshold equals to zero.

This value is independent to \( \alpha \) and \( \sigma \) when \( P(X = -1) = P(X = 1) = 0.5 \).

Following is an example:

In this question, due to the same probability for \( X = 1 \) and \( X = -1 \), the value of \( \sigma \) doesn’t matter.