Review of Last Time

• |= means “logically follows”
• |-i means “can be derived from”
• If your inference algorithm derives only things that follow logically from the KB, the inference is sound
• If everything that follows logically from the KB can be derived using your inference algorithm, the inference is complete
Entailment Applied to the Wumpus World

Let us consider the models that support the conclusion $\alpha_1 = \text{"There is no pit in [1,2]."}$ We draw a line marked with $\alpha_1$ around these models. In every model in which $\text{KB}$ is true, $\alpha_1$ is also true. Therefore $\text{KB} \models \alpha_1$

Inference: Model Checking

- Suppose we want to know if $\text{KB} \models \neg P_{1,2}$?
- In the 3 models in which $\text{KB}$ is true, $\neg P_{1,2}$ is also true

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Complexity

• If the KB and $\alpha$ contain n symbols in total, what is the time complexity of the truth table enumeration algorithm?

• Space complexity is $O(n)$ because the actual algorithm uses DFS

The really depressing news

• Every known inference algorithm for propositional logic has a **worst-case** complexity that is **exponential** in the size of the input

• But some algorithms are more efficient **in practice**
Logical equivalence

• Intuitively: two sentences $\alpha$ and $\beta$ are logically equivalent (i.e. $\alpha \equiv \beta$) if they are true in the same set of models

• Formally: $\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$

• Can prove this with truth tables

Standard Logic Equivalences

\[
\begin{align*}
(\alpha \land \beta) & \equiv (\beta \land \alpha) \quad \text{commutativity of $\land$} \\
(\alpha \lor \beta) & \equiv (\beta \lor \alpha) \quad \text{commutativity of $\lor$} \\
((\alpha \land \beta) \land \gamma) & \equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of $\land$} \\
((\alpha \lor \beta) \lor \gamma) & \equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of $\lor$} \\
\neg(\neg\alpha) & \equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv (\neg\alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \iff \beta) & \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\neg\alpha \land \beta) & \equiv (\neg\alpha \lor \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \lor \beta) & \equiv (\neg\alpha \land \neg\beta) \quad \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) & \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of $\land$ over $\lor$} \\
(\alpha \lor (\beta \land \gamma)) & \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of $\lor$ over $\land$}
\end{align*}
\]

In the above, $\alpha$, $\beta$, and $\gamma$ are arbitrary sentences of propositional logic
Validity

- A sentence is valid if it is true in all models
- E.g. $P \lor \neg P$ is valid
- Valid sentences = Tautologies
- Tautologies are vacuous

Deduction theorem
For any sentences $\alpha$ and $\beta$, $\alpha \models \beta$ iff the sentence $(\alpha \Rightarrow \beta)$ is valid

Satisfiability

- A sentence is satisfiable if it is true in some model.
- A sentence is unsatisfiable if it is true in no models
- Determining the satisfiability of sentences in propositional logic was the first problem proved to be NP-complete
- Satisfiability is connected to validity:
  $\alpha$ is valid iff $\neg \alpha$ is unsatisfiable
- Satisfiability is connected to entailment:
  $\alpha \models \beta$ iff the sentence $(\alpha \land \neg \beta)$ is unsatisfiable (proof by contradiction)
Exercise

• Is the following sentence valid?

\[(A \Rightarrow B) \lor (\neg A \Rightarrow \neg B)\]

Proof methods

How do we prove that \(\alpha\) can be entailed from the KB?
1. Model checking e.g. check that \(\alpha\) is true in all models in which KB is true
2. Inference rules
Inference Rules

1. Modus Ponens

\[
\alpha \Rightarrow \beta, \quad \alpha \\
\beta
\]

2. And-Elimination

\[
\alpha \land \beta \\
\alpha
\]

These are both sound inference rules. You don’t need to enumerate models now.

Other Inference Rules

\[
(\alpha \land \beta) \equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) \equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) \equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) \equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg (\neg \alpha) \equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) \equiv (\neg \beta \Rightarrow \neg \alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) \equiv (\neg \alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg (\alpha \land \beta) \equiv (\neg \alpha \lor \neg \beta) \quad \text{de Morgan} \\
\neg (\alpha \lor \beta) \equiv (\neg \alpha \land \neg \beta) \quad \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\]

All of the logical equivalences can be turned into inference rules e.g.

\[
\alpha \Leftrightarrow \beta \\
(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)
\]
Example

Given the following KB, can we prove $\neg R$?

KB:

\[
P \Rightarrow \neg (Q \lor R) \\
P
\]

Proof:

$\neg (Q \lor R)$ by Modus Ponens

$\neg Q \land \neg R$ by De Morgan’s Law

$\neg R$ by And-Elimination

Proofs

- A sequence of applications of inference rules is called a proof

- Instead of enumerating models, we can search for proofs

- Proofs ignore irrelevant propositions

- 2 methods:
  - Go forward from initial KB, applying inference rules to get to the goal sentence
  - Go backward from goal sentence to get to the KB
In-class Exercise

<table>
<thead>
<tr>
<th>Statement</th>
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<tbody>
<tr>
<td>If it is October, there will not be a football game at OSU</td>
<td></td>
</tr>
<tr>
<td>If it is October and it is Saturday, I will be in Corvallis</td>
<td></td>
</tr>
<tr>
<td>If it doesn’t rain or if there is a football game, I will ride my bike to OSU</td>
<td></td>
</tr>
<tr>
<td>Today is Saturday and it is October</td>
<td></td>
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<tr>
<td>If I am in Corvallis, it will not rain</td>
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</tbody>
</table>

Can you prove that I will ride my bike to OSU?

Monotonicity

- Proofs only work because of monotonicity
- Monotonicity: the set of entailed sentences can only increase as information is added to the knowledge base
- For any sentences $\alpha$ and $\beta$, if $\text{KB} \models \alpha$ then $\text{KB} \land \beta \models \alpha$
Resolution

- An inference rule that is sound and complete
- Forms the basis for a family of complete inference procedures
- Here, complete means refutation completeness: resolution can refute or confirm the truth of any sentence with respect to the KB

Resolution

- Here’s how resolution works ($\neg l_2$ and $l_2$ are called complementary literals):

\[
\frac{l_1 \lor l_2, \quad \neg l_2 \lor l_3}{l_1 \lor l_3}
\]

- Note that you need to remove multiple copies of literals (called factoring) i.e.

\[
\frac{l_1 \lor l_2, \quad \neg l_2 \lor l_1}{l_1}
\]

- If $l_i$ and $m_j$ are complementary literals, the full resolution rule looks like:

\[
\frac{l_1 \lor \cdots \lor l_k, \quad m_1 \lor \cdots \lor m_n}{l_1 \lor \cdots \lor l_{i-1} \lor l_{i+1} \lor \cdots \lor l_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n}
\]
In-class Exercise

Can we show that:

KB |= (Socrates ⇒ Mortal)?

 Conjunctive Normal Form

- Resolution only applies to sentences of the form \( l_1 \lor l_2 \lor \ldots \lor l_k \)
- This is called a disjunction of literals
- It turns out that every sentence of propositional logic is logically equivalent to a conjunction of disjunction of literals
- Called Conjunctive Normal Form or CNF
  e.g. \((l_1 \lor l_2 \lor l_3 \lor l_4) \land (l_5 \lor l_6 \lor l_7 \lor l_8) \land \ldots\)
- \(k\)-CNF sentences have exactly \(k\) literals per clause
  e.g. A 3-CNФ sentence would be \((l_1 \lor l_2 \lor l_3) \land (l_4 \lor l_5 \lor l_6) \land (l_7 \lor l_8 \lor l_9)\)
Recipe for Converting to CNF

1. Eliminate $\leftrightarrow$, replacing $\alpha \leftrightarrow \beta$ with $(\alpha \implies \beta) \land (\beta \implies \alpha)$
2. Eliminate $\implies$, replacing $\alpha \implies \beta$ with $\neg \alpha \lor \beta$
3. Move $\neg$ inwards using:
   - $\neg(\neg \alpha) \equiv \alpha$ (double-negation elimination)
   - $\neg(\alpha \land \beta) \equiv \neg \alpha \lor \neg \beta$ (De Morgan’s Law)
   - $\neg(\alpha \lor \beta) \equiv \neg \alpha \land \neg \beta$ (De Morgan’s Law)
4. Apply distributive law $(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$

Exercise

- Convert the following sentence to CNF.

  $(B \lor C) \implies D$
A resolution algorithm

To prove $KB \models \alpha$, we show that $(KB \land \neg \alpha)$ is unsatisfiable

(Remember that $\alpha \models \beta$ iff the sentence $(\alpha \land \neg \beta)$ is unsatisfiable)

The algorithm:
1. Convert $(KB \land \neg \alpha)$ to CNF
2. Apply resolution rule to resulting clauses. Each pair with complementary literals is resolved to produce a new clause which is added to the KB
3. Keep going until
   - There are no new clauses that can be added (meaning $KB \models \alpha$)
   - Two clauses resolve to yield the empty clause (meaning $KB \models \alpha$)

The empty clause is equivalent to false because a disjunction is true only if one of its disjuncts is true

Exercise

- Suppose the KB contains the following sentences in CNF.
  1. $\neg C \lor E$
  2. $\neg P \lor E$
  3. $\neg E \lor \neg R$
  4. $\neg A \lor \neg P \lor E$
- Does $KB \models \neg R$?
Resolution Pseudocode

```plaintext
function PL-Resolution(KB, α) returns true or false
    clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
    new ← { }
    loop do
        for each C_i, C_j in clauses do
            resolvents ← PL-Resolve(C_i, C_j)
            if resolvents contains the empty clause then return true
            new ← new ∪ resolvents
        if new ⊆ clauses then return false
        clauses ← clauses ∪ new
    end loop
    return false
```

Things you should know

- Understand the syntax and semantics of propositional logic
- Know how to do a proof in propositional logic using inference rules
- Know how to convert arbitrary sentences to CNF
- Know how resolution works