Problem 1.
Consider the experiment of placing two distinguishable particles \( a \) and \( b \) into three cells. Examples of sample points are \( \omega_1 = (ab|\cdot|\cdot) \), \( \omega_2 = (a|b|\cdot) \) etc.

(a) Write down the sample space \( \Omega \).
\[
\Omega = \begin{cases} 
(a | b | \cdot), (b | a | \cdot), (a | \cdot | b), (b | \cdot | a), (- | a | b), (- | b | a), \\
(ab | \cdot | \cdot), (- | ab | \cdot), (- | \cdot | ab) 
\end{cases}
\]

(b) Define events as:
- A: multiple particles occupy a cell
- B: At least two cells are occupied
- C: A and B both occurs

Identify the sample points in events A, B and C. Assuming equal probability for each outcome, calculate the probability of events A, B and C.

\( A = \{(ab | \cdot | \cdot), (- | ab | \cdot), (- | \cdot | ab)\} \)
\( P(A) = \frac{3}{9} = \frac{1}{3} \)

\( B = \{(a | b | \cdot), (b | a | \cdot), (a | \cdot | b), (b | \cdot | a), (- | a | b), (- | b | a), (- | a | b), (- | b | a)\} \)
\( P(B) = \frac{6}{9} = \frac{2}{3} \)

\( C = \{\} \) = empty set
\( P(C) = 0 \)

c) Suppose these three particles were indistinguishable. Examples of sample points are \( \omega'_1 = (** | \cdot | \cdot) \), \( \omega'_2 = (\cdot | ** | \cdot) \) etc. Write down the new sample space \( \Omega' \).
\[
\Omega' = \begin{cases} 
(** | * | \cdot), (* | \cdot | **), (- | * | *) \\
(* * | \cdot | \cdot), (- | ** | \cdot), (- | \cdot | **) 
\end{cases}
\]

Problem 2.
An experiment consists of tossing two six sided dice.

(a) Find the sample space \( S \).
Since,

Let $A, B$ and $C$ be events in sample space $S$.

**Problem 3.**

b) Find the probability of event $A$ that the sum of the dots on the dice equals 5.

$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$P(A) = \frac{4}{36} = \frac{1}{9}$

c) Find the probability of event $B$ that the sum of the dots on the dice is greater than 10.

$B = \{(5, 6), (6, 5), (6, 6)\}$

$P(A) = \frac{3}{36} = \frac{1}{12}$

d) Find the probability of event $C$ that the sum of the dots on the dice is greater than 7 but less than 12.

$C = \{(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5)\}$

$P(C) = \frac{14}{36} = \frac{7}{18}$

e) Calculate $P(A \cup B \cup C)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$A \cap B = \emptyset \rightarrow P(A \cap B) = 0$

$A \cap C = \emptyset \rightarrow P(A \cap C) = 0$

$B \cap C = \emptyset \rightarrow P(B \cap C) = \frac{2}{36}$

$A \cap B \cap C = \emptyset \rightarrow P(A \cap B \cap C) = 0$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{4}{36} + \frac{3}{36} + \frac{14}{36} - \frac{2}{36} = \frac{19}{36}$

**Problem 3.**

Let $A$, $B$ and $C$ be events in sample space $S$.

a) Using axioms and laws of probability prove that

i) $P(A \cap B \cap C) \leq P(A \cup B \cup C)$

**Solution 1:** $(A \cap B \cap C) \subseteq (A \cup B \cup C) \rightarrow P(A \cap B \cap C) \leq P(A \cup B \cup C)$

**Solution 2:**

$P(A \cup B \cup C) = P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + (P(B) + P(C) - P(B \cap C)) - P((A \cap B) \cup (A \cap C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - (P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

Since,

$P(A) - P(A \cap B) \geq 0$
\[ P(B) - P(B \cap C) \geq 0 \]
\[ P(C) - P(A \cap C) \geq 0 \]
this implies that,
\[ P(A \cup B \cup C) \geq P(A \cap B \cap C) \]

\[ \text{ii) } P(A \cup B \cup C) \leq P(A) + P(B) + P(C) \]
\[ P(A \cup B \cup C) = P(A \cup (B \cup C)) \]
\[ = P(A) + P(B \cup C) - P(A \cap (B \cup C)) \]
\[ = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B \cup C) \]
Since,
\[ P(B \cap C) + P(A \cap (B \cup C)) \geq 0, \]
this implies that, \[ P(A \cup B \cup C) \leq P(A) + P(B) + P(C) \]

b) When does the equality holds in the above expressions i) and ii)? Illustrate using Venn diagrams.

i) \( A = B = C \)

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$S_7 = \{0, 14, 28, \ldots, 196\}$
Number of elements in set $S_5 = 1 + \left\lfloor \frac{200}{2 \times 7} \right\rfloor = 1 + 14 = 15$

We need to find the probability of event $\bar{S}$, that the number selected from $S$ is divisible by both 2 and 5 but not by 7. This implies that the number has to be selected from the gray colored region in the Venn diagram.

$S = S_2$

$$P(\bar{S}) = P(S_2 \cap S_5) - P(S_2 \cap S_5 \cap S_7)$$
$$= P(S_5) - P(S_5 \cap S_7)$$
$$= \frac{21}{101} - P(S_5 \cap S_7)$$

$S_5 \cap S_7 = \{0, 70, 140\}$
Number of elements in set $(S_5 \cap S_7) = 1 + \left\lfloor \frac{200}{2 \times 5 \times 7} \right\rfloor = 1 + 2 = 3$
Therefore, $P(S_5 \cap S_7) = \frac{3}{101}$

Therefore the probability that the number selected from $S$ is divisible by both 2 and 5 but not from 7 is equal to,
$$P(\bar{S}) = \frac{21}{101} - \frac{3}{101} = \frac{18}{101}$$