1 Problem 1

(a) Calculate the probability that a car at parking spot \( k \) will be detected by camera. Let \( \theta_1 \) and \( \theta_2 \) be the camera angle at beginning and the end of parking lot \( k \), respectively. We have \( \theta_1 = \tan^{-1}\left(\frac{3(k-1)}{10}\right) \) and \( \theta_2 = \tan^{-1}\left(\frac{3k}{10}\right) \).

A car is detected by the camera if it lies in a parking lot or its angle is within \([\theta_1, \theta_2]\).

\[
P(\theta_1 \leq \theta \leq \theta_2) = \frac{\theta_2 - \theta_1}{\pi} = \frac{\theta_2 - \theta_1}{\pi} = \frac{\tan^{-1}\left(\frac{3k}{10}\right) - \tan^{-1}\left(\frac{3(k-1)}{10}\right)}{\pi} \text{ (By Uniform distribution).}
\]

(b) Parking spot 1 has highest detection probability. This is because the probability of detecting the car is \( p = \frac{\theta_2 - \theta_1}{\pi} \) and when the parking lots are far from the beginning position (at 0), the differences between the angles are smaller (See prove).

Prove: See the figure below: The problem is equivalent to: Prove that \( \angle A_1 > \angle A_2 > \angle A_3 \).

For \( \angle A_1 > \angle A_2 \) prove: Draw BN and DM perpendicularly to AC. It is easy to see \( \triangle BNC = \)
\( \triangle DMC \), therefore \( BN = DM \).

We also have: \( \sin(\angle A_1) = \frac{BN}{AB} \) and \( \sin(\angle A_2) = \frac{DM}{AD} \).

Since \( BN = DM \) and \( AB < AD \), \( \sin(\angle A_1) > \sin(\angle A_2) \), or \( \angle A_1 > \angle A_2 \).

Do the similar prove for other angles differences.

(c) From 9:59pm to 6:00am, there are 16 different times the camera activated.

The probability of the 2nd car is detected by the camera at each time:

\[
p = P(\text{2nd car is detected}) = \frac{\theta_2 - \theta_1}{\pi} = \frac{\tan^{-1}\left(\frac{6}{10}\right) - \tan^{-1}\left(\frac{3}{10}\right)}{\pi} = 0.0792 = 7.92%.
\]

The probability that the missing car will be detected by the camera by 6:00am tomorrow:

\[
P(\text{missing car is detected by 6:00am}) = \sum_{k=1}^{16} (1 - p)^{(k-1)}p
= 1 - \sum_{k=17}^{\infty} (1 - p)^{(k-1)}p
= 1 - (1 - p)^{16}\left(1 + (1 - p) + (1 - p)^2 + \ldots\right)p
= 1 - (1 - p)^{16} \frac{1}{p}
= 1 - (1 - 0.0792)^{16}
= 0.7329 = 73.29%.
\]

(d) To guarantee a probability of detecting the missing car by 6:00am at least 95%, how often should the camera be used instead of half an hour?

Let \( N \) be the number of different times the camera activated from 9:59pm to 6:00am.

The probability that the missing car will be detected by the camera by 6:00am is:

\[
P(\text{missing car is detected by 6:00am}) = 1 - (1 - p)^N
= 1 - (1 - 0.0792)^N
\]

To guarantee the probability at least 95%, we have:

\[
P(\text{missing car is detected by 6:00am}) = 1 - (1 - 0.0792)^N \leq 95%
\]

\( \Leftrightarrow N \geq 36.3064 \)

Therefore, we need more than 36 different times or less than \( \frac{8 \text{ hours detecting}}{36 \text{ times}} = 13.3 \) minutes from 9:59pm to 6:00am to guarantee that the probability at least 95%.
2 Problem 2

Let $X$ be the number of bits received at the receiver.
We have: $\lambda = 10$ bits per $1\mu$s, so in general, $\lambda_T = \lambda T = 10T$.

(a) The probability that the receiver receives more than 13 bits in $1\mu$s:

$$P(X > 13) = \sum_{k=14}^{\infty} \frac{\lambda_T^k e^{-\lambda_T}}{k!}$$

$$= 1 - \sum_{k=0}^{13} \frac{\lambda_T^k e^{-\lambda_T}}{k!}$$

$$= 1 - \sum_{k=0}^{13} \frac{10^k e^{-10}}{k!}$$

$$= 0.1355 \text{ (By MATLAB)}.$$

(b) The probability that the receiver receives more than 130 bits in $10\mu$s:

$$P(X > 130) = \sum_{k=131}^{\infty} \frac{\lambda_T^k e^{-\lambda_T}}{k!}$$

$$= 1 - \sum_{k=0}^{130} \frac{\lambda_T^k e^{-\lambda_T}}{k!}$$

$$= 1 - \sum_{k=0}^{130} \frac{100^k e^{-100}}{k!}$$

$$= 0.0017 \text{ (By MATLAB)}.$$

(c) MATLAB code:

```matlab
%% HOMEWORK 5 - Solution 

% Question (a)
k = 0:13;
% Solution 1
p1 = 1 - sum(10.^k ./ factorial(k) * exp(-10));
% Solution 2
ex = k * log(10) - gammaln(k+1) - 10;
p2 = 1 - sum(exp(ex));

% Question (b)
k = 0:130;
ex = 2 * k * log(10) - gammaln(k+1) - 1e2;
p3 = 1 - sum(exp(ex));
```
3 Problem 3

The mouse will be trapped when it goes to the left or right branches. We have:

\[
P(k = 1) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}
\]
\[
P(k = 2) = \frac{2}{3} \times \left( \frac{1}{12} + \frac{1}{4} \right) = \frac{2}{3} \times \frac{1}{3}
\]
\[
P(k = 3) = \frac{2}{3} \times \frac{2}{3} \times \left( \frac{1}{12} + \frac{1}{4} \right) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}
\]
\[
P(k = 4) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \left( \frac{1}{12} + \frac{1}{4} \right) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}
\]
\[\vdots\]
\[
P(k = n) = \left( \frac{2}{3} \right)^{n-1} \left( \frac{1}{12} + \frac{1}{4} \right) = \left( \frac{2}{3} \right)^{n-1} \times \frac{1}{3}
\]

Therefore, the probability of the mouse being trapped at level \( k = n \):

\[
P(k = n) = \begin{cases} 
\frac{2^{n-1}}{3^n}, & \text{for } n > 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Or \( P(k = n) = \text{Geometric}(p = \frac{1}{3}) \).

4 Problem 4

The set of resistors have 5% tolerance; or the acceptable resistance range of resistors is:

\[
1000 \pm (1000 \times 5\%) = 1000 \pm 50
\]

or: \( 950 \Omega \leq X \leq 1050 \Omega \).

The probability of \( X \) in the acceptable range:

\[
P(950 \leq X \leq 1050) = F_X(1050) - F_X(950^-)
\]
\[
= \Phi\left( \frac{1050 - 1000}{25} \right) - \Phi\left( \frac{950 - 1000}{25} \right)
\]
\[
= \Phi(2) - \Phi(-2)
\]
\[
= \Phi(2) - (1 - \Phi(2))
\]
\[
= 2\Phi(2) - 1.
\]

The probability that a resistor picked will be rejected:

\[
P\left( \{X < 950\} \cup \{X > 1050\} \right) = 1 - P(950 \leq X \leq 1050)
\]
\[
= 1 - \left( 2\Phi(2) - 1 \right)
\]
\[
= 2 - 2\Phi(2)
\]
\[
= 0.0456.
\]