ECE 353: Introduction to Probability and Random Signals - Spring 2018
Homework 4 - Solution

Problem 1.
Random variable Y has a probability mass function (pmf) as \( p_Y(y) = \begin{cases} \frac{c}{y} & , y = 1, 2, 3 \\ \frac{c}{y^2} & , y = -1, -2, -3 \\ 0 & , \text{otherwise} \end{cases} \)

a) Find the value of the constant \( c \)

Range of \( Y \): \( \mathcal{R}_Y = \{-1, -2, -3, 1, 2, 3\} \)

\[ \sum_{y \in \mathcal{R}_Y} p_Y(y) = 1 \]

\[ p_Y(y = 1) + p_Y(y = 2) + p_Y(y = 3) + p_Y(y = -1) + p_Y(y = -2) + p_Y(y = -3) = 1 \]

\[ \frac{c}{1} + \frac{c}{2} + \frac{c}{3} + \frac{c}{(-1)^2} + \frac{c}{(-2)^2} + \frac{c}{(-3)^2} = 1 \]

\[ c \left( 1 + \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{4} + \frac{1}{9} \right) = 1 \]

\[ c = \frac{36}{115} \]

b) Now that the constant \( c \) is determined, find

(i) Probability of \( Y = 1 \)

\[ p_Y(Y = 1) = \frac{36}{115} \]

(ii) Probability of \( Y < 1 \)

\[ p_Y(Y < 1) = p_Y(\ y = -1) + p_Y(\ y = -2) + p_Y(\ y = -3) + p_Y(\ y < 3) \]

\[ = \frac{36}{115 \times (-1)^2} + \frac{36}{115 \times (-2)^2} + \frac{36}{115 \times (-3)^2} + 0 \]

\[ = \frac{36}{115} \left( 1 + \frac{1}{4} + \frac{1}{9} \right) \]

\[ = \frac{36}{115} \times \frac{49}{36} \]

\[ = \frac{49}{115} \]
**Problem 2.**
The number of hits at a website in a time interval has the pmf of following form,

\[ p_H(h) = \begin{cases} \frac{\alpha^h e^{-\alpha}}{h!}, & h \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

where \( \alpha \) is the average number of hits in a time interval.

a) Given that \( \alpha = 0.5 \), find the probability that there are no hits in the time interval.

The pmf is,

\[ p_H(h = 0) = \frac{0.5^0 e^{-0.5}}{0!} = e^{-0.5} \]

b) Given that \( \alpha = 2 \), what is the probability that there are no more than two hits in the time interval.

The pmf is,

\[ p_H(h \leq 2) = p_H(h = 0) + p_H(h = 1) + p_H(h = 2) \]

\[ = \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} \]

\[ = e^{-2} + 2e^{-2} + 2e^{-2} = 5e^{-2} \]

**Problem 3.**
A probability density function (pdf) of a random variable \( X \) is given by:

\[ f_X(x) = \begin{cases} c(x + 2), & x \in (-2, 2) \\ 0, & \text{otherwise} \end{cases} \]

a) Given that \( c \) is positive, find \( c \).

\[ \int_{-\infty}^{\infty} f_X(x)dx = 1 \]

\[ \int_{-2}^{2} c(x + 2)dx = 1 \]

\[ c(\frac{x^2}{2} + 2x)|_{-2}^2 = 1 \]

\[ 8c = 1 \]

\[ c = \frac{1}{8} \]

b) Calculate the cumulative distribution function (CDF) of \( X \).
\[ x \leq -2 \quad ; \quad F_X(x) = 0 \]

\[-2 < x < 2 \quad ; \quad F_X(x) = \int_{-2}^{x} \left( \frac{1}{8}(x + 2) \right) \, dx = \frac{1}{8}(\frac{x^2}{2} + 2x) \bigg|_{-2}^{x} = \frac{1}{8}(\frac{x^2}{2} + 2x - 2 + 4) = \frac{1}{8}(\frac{x^2}{2} + 2x + 2) = \frac{1}{16}(x + 2)^2 \]

\[ x \geq 2 \quad ; \quad F_X(x) = 1 \]

**Problem 4.** In a restaurant known for its unusual service, the time, in minutes, that a customer has to wait before he captures the attention of a waiter is specified by the following CDF.

\[
F_X(x) = \begin{cases} 
\frac{x^2}{36}, & 0 \leq x \leq 3 \\
\frac{1}{4}x - \frac{1}{2}, & 3 < x \leq 5 \\
\frac{3}{4}, & 5 < x \leq 6 \\
\frac{1}{8}x, & 6 < x \leq 8 \\
1, & x > 8
\end{cases}
\]

a) Compute and sketch the PDF \( f_X(x) \)

\[ x \leq 0 \quad ; \quad f_X(x) = 0 \]

\[ 0 \leq x \leq 3 \quad ; \quad f_X(x) = \frac{d}{dx} \left( \frac{x^2}{36} \right) = \frac{x}{18} \]

\[ 3 < x \leq 5 \quad ; \quad f_X(x) = \frac{d}{dx} \left( \frac{1}{4}x - \frac{1}{2} \right) = \frac{1}{4} \]

\[ 5 < x \leq 6 \quad ; \quad f_X(x) = \frac{d}{dx} \left( \frac{3}{4} \right) = 0 \]

\[ 6 < x \leq 8 \quad ; \quad f_X(x) = \frac{d}{dx} \left( \frac{1}{8}x \right) = \frac{1}{8} \]

\[ x > 8 \quad ; \quad f_X(x) = 0 \]
b) Verify the area under the PDF is indeed unity

Area under the PDF = \( \frac{1}{2} \times \frac{3}{18} + \frac{1}{4} \times 2 + 0 \times 1 + \frac{1}{8} \times 2 = 1 \)

c) What is the probability that the customer will have to wait greater than 2 minutes but less than 7 minutes?

\[
f_X(2 < x < 7) = f_X(\{x < 7\}) - f_X(\{x \leq 2\}) = F_X(x = 7^{-}) - F_X(x = 2) = \frac{7}{8} - \frac{2^2}{36} = \frac{7}{8} - \frac{1}{9} = \frac{55}{72} = 0.7638
\]