CS 331: Artificial Intelligence
Adversarial Search

Games we will consider

• Deterministic
• Discrete states and decisions
• Finite number of states and decisions
• Perfect information i.e. fully observable
• Two agents whose actions alternate
• Their utility values at the end of the game are equal and opposite (we call this zero-sum)

“it's not enough for me to win, I have to see my opponents lose"
Nim

Formal Definition of Nim
A quintuplet \((S, I, \text{Succ}(), T, U)\):

\[ S = \text{Max}(\text{III}), \text{Max}(\text{II}), \text{Max}(\text{I}) \]

\[ \text{Min}(\text{III}), \text{Min}(\text{II}), \text{Min}(\text{I}) \]

\[ \text{I} = \text{Max}(\text{III}) \]

\[ \text{Succ}() = \begin{cases} \text{Succ}(\text{Max}(\text{III})) = \{\text{Min}(\text{III}), \text{Min}(\text{II}), \text{Min}(\text{I})\} & \text{Max}(\text{III}) \rightarrow \text{Min}(\text{III}) \\ \text{Succ}(\text{Max}(\text{II})) = \{\text{Min}(\text{II}), \text{Min}(\text{I})\} & \text{Max}(\text{II}) \rightarrow \text{Min}(\text{II}) \\ \text{Succ}(\text{Max}(\text{I})) = \{\text{Min}(\text{I})\} & \text{Max}(\text{I}) \rightarrow \text{Min}(\text{I}) \\ \end{cases} \]

\[ T = \text{Max}(\text{I}), \text{Max}(\text{II}), \text{Max}(\text{III}), \text{Min}(\text{I}), \text{Min}(\text{II}), \text{Min}(\text{III}) \]

\[ U = \text{Utility}(\text{Max}(\text{I}) \text{ or } \text{Max}(\text{II}) \text{ or } \text{Max}(\text{III})) = +1, \]

\[ \text{Utility}(\text{Min}(\text{I}) \text{ or } \text{Min}(\text{II}) \text{ or } \text{Min}(\text{III})) = -1 \]

Nim Game Tree

We'll call the players Max and Min, with Max starting first

How to Use a Game Tree

- Max wants to maximize his utility
- Min wants to minimize Max’s utility
- Max’s strategy must take into account what Min does since they alternate moves
- A move by Max or Min is called a ply

The Minimax Value of a Node

The minimax value of a node is the utility for MAX of being in the corresponding state, assuming that both players play optimally from there to the end of the game

\[ \text{MINIMAX - VALUE}(n) = \begin{cases} \text{UTILITY}(n) & \text{if } n \text{ is a terminal state} \\ \max_{s \in \text{Successors}(n)} \text{MINIMAX - VALUE}(s) & \text{if } n \text{ is a MAX node} \\ \min_{s \in \text{Successors}(n)} \text{MINIMAX - VALUE}(s) & \text{if } n \text{ is a MIN node} \end{cases} \]

Minimax value maximizes worst-case outcome for MAX
Minimax Values in Nim Game Tree

Minimax decision at the root: taking this action results in the successor with highest minimax value.
The MINIMAX algorithm

- Computes minimax decision from the current state
- Depth-first exploration of the game tree
- Time Complexity $O(b^m)$ where $b$=# of legal moves, $m$=maximum depth of tree
- Space Complexity:
  - $O(bm)$ if all successors generated at once
  - $O(m)$ if only one successor generated at a time (each partially expanded node remembers which successor to generate next)

The MINIMAX Algorithm

Function `MINIMAX-DECISION(state)` returns an action
inputs: state, current state in game
\[ v \leftarrow \text{MAX-VALUE(state)} \]
return the action in `SUCCESSORS(state)` with value $v$

Function `MAX-VALUE(state)` returns a utility value
if `TERMINAL-TEST(state)` then return `UTILITY(state)`
\[ v \leftarrow -\infty \]
for $a, s$ in `SUCCESSORS(state)` do
  \[ v \leftarrow \text{MAX}(v, \text{MIN-VALUE}(s)) \]
return $v$

Function `MIN-VALUE(state)` returns a utility value
if `TERMINAL-TEST(state)` then return `UTILITY(state)`
\[ v \leftarrow \infty \]
for $a, s$ in `SUCCESSORS(state)` do
  \[ v \leftarrow \text{MIN}(v, \text{MAX-VALUE}(s)) \]
return $v$

Minimax With 3 Players

Now have a vector of utilities for players (A,B,C). All players maximize their utilities. Note: In two-player, zero-sum games, we have a single value because the values are always opposite.
Minimax With 3 Players

A
B
C (1,2,6) (6,1,2) (1,5,2) (5,4,5)
(1,2,6) (4,2,3) (6,1,2) (7,4,1) (5,1,1) (1,5,2) (7,7,1) (5,4,5)

Minimax With 3 Players

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Subtleties With Multiplayer Games

• Alliances can be made and broken
  • For example, if A and B are weaker than C, they can gang up on C
  • But A and B can turn on each other once C is weakened
  • But society considers the player that breaks the alliance to be dishonorable

Pruning

• Can we improve on the time complexity of $O(b^n)$?
  • Yes if we prune away branches that cannot possibly influence the final decision

Pruning in Nim

Max
Min
Max
Min
Max
Min
Max
Min
Max
Min
Max
Min

If we know that the only two outcomes are +1 and -1, what branches do we not need to explore when minimax backtracks?
Pruning in Nim

If we know that the only two outcomes are +1 and -1, what branches do we not need to explore when minimax backtracks?

Pruning Intuition (General Case)

The max player will never choose the right subtree once it knows that it is upper bounded by 1.

Suppose we just went down this branch. We know that the minimax value of its parent will be ≤ 1.

Pruning Example

MINIMAX-VALUE(root) = max(min(3,12,8),min(2,x,y),min(14,5,2)) = max(3,min(2,x,y),2) = max(3,z) where z ≤ 2 = 3

Pruning Intuition

Remember that minimax search is DFS. At any one time, we only have to consider the nodes along a single path in the tree.

In general, let:
• α = highest minimax value of all of the MAX player’s choices expanded on current path (best score for MAX so far)
• β = lowest minimax value of all of the MIN player’s choices expanded on current path (best score for MIN so far)
• If at a MIN player node, prune if minimax value of node ≤ α
• If at a MAX player node, prune if minimax value of node ≥ β

ALPHA-BETA Pseudocode

function ALPHA-BETA-SEARCH(state) returns an action
inputs: state, current state in game
v ← MAX-VALUE(state, -∞, +∞)
return the action in SUCCESSORS(state) with value v

function MAX-VALUE(state, α, β) returns a utility value
inputs: state, current state in game
α, the value of the best alternative for MAX along the path to state
β, the value of the best alternative for MIN along the path to state
if TERMINAL-TEST(state) then return UTILITY(state)
v ← -∞
for a, s in SUCCESSORS(state) do
r ← MAX(β, MIN-VALUE(s, α, β))
if r ≥ β then return v
α ← MAX(a, v)
return v
**Alpha-Beta Pruning Example**

- **a)** $(-\infty, +\infty)$
- **b)** $(-\infty, +\infty)$
- **c)** $(-\infty, +\infty)$
- **d)** $(-\infty, +\infty)$
- **e)** $(-\infty, +\infty)$
- **f)** $(-\infty, +\infty)$
- **g)** $(-\infty, +\infty)$
- **h)** $(-\infty, +\infty)$

**Effectiveness of Alpha-Beta**

- Depends on order of successors
- Best case: Alpha-Beta reduces complexity from $O(b^m)$ for minimax to $O(b^{m/2})$
- This means Alpha-Beta can lookahead about twice as far as minimax in the same amount of time

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**Illustrating the Pseudocode**

- In the example to follow, the notation $(-\infty, +\infty)$ represents the ($\alpha$, $\beta$) values for the corresponding node
- This example is intended to illustrate how the actual implementation of Alpha-Beta pruning works

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Implementation Details

• In games we have the problem of **transposition**
• Transposition means different permutations of the move sequence that end up in the same position
• Results in lots of repeated states
• Use a transposition table to remember the states you’ve seen (similar to closed list)

What you should know

• Be able to draw up a game tree
• Know how the Minimax algorithm works
• Know how the Alpha-Beta algorithm works
• Be able to do both algorithms by hand