# CS 331: Artificial Intelligence
# Game Theory III

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## Continuous Action Spaces

- Previously, we only allowed the players to choose from a finite set of actions
- Today, we'll see how to calculate Nash Equilibria when we have a continuous action space

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# Tragedy of the Commons (Hardin 1968)

- Illustrates the conflict for resources between individual interests and the common good
- If citizens respond only to private incentives, public goods will be underprovided and public resources overutilized



# Tragedy of the Commons

- n farmers in a village graze goats on the commons to eventually fatten and sell
- The more goats they graze the less well fed they are
- And so the less money they get when they sell them

# Tragedy of the Commons (Formalized)

- n farmers
- $g_i$ goats allowed to graze on the commons by the *i*th farmer
- Assume goats are continuously divisible ie.

  $g_i \in [0, 36]$
- Total number of goats in the village is

  $G = g_1 + \ldots + g_n$.
- Strategy profile $(g_1, g_2, \ldots, g_n)$.

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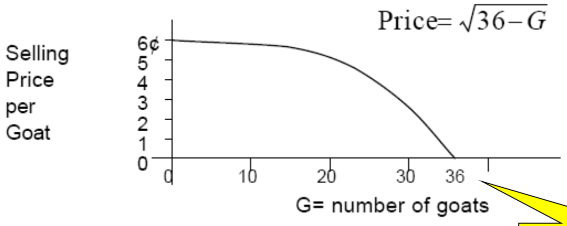
# Payoff for Goats




Note: Price per goat = 0 if G > 36

Payoff for farmer i

= Price per goat * # of goats

$$= g_i \sqrt{36 - G}$$

$$= g_i \sqrt{36 - \sum_{j=1}^{N} g_j}$$

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# Calculating the Nash Equilibrium

- Suppose a Nash Equilibrium exists using the strategy profile ($g_1^*$, $g_2^*$, …, $g_n^*$)
- This means that

$$g_i^* = \arg\max_{g_i} \left[ \begin{array}{l} \text{Payoff to farmer i assuming} \\ \text{the other players play} \\ (g_1^*, g_2^*, \ldots, g_n^*) \end{array} \right]$$

- Define $G_{-i}^* = \sum_{j \neq i} g_j^*$
- Therefore $g_i^* = \arg\max_{g_i} \left( g_i \sqrt{36 - g_i - G_{-i}^*} \right)$
- Use calculus to compute $g_i^*$!



# Calculating the Nash Equilibrium

$$\frac{\partial}{\partial g_i^*} g_i^* \sqrt{36 - g_i^* - G_{-i}^*} = 0$$

$$\Rightarrow \left( \frac{\partial}{\partial g_i^*} g_i^* \right) \sqrt{36 - g_i^* - G_{-i}^*} + g_i^* \left( \frac{\partial}{\partial g_i^*} \sqrt{36 - g_i^* - G_{-i}^*} \right) = 0$$

$$\Rightarrow \sqrt{36 - g_i^* - G_{-i}^*} - \frac{g_i^*}{2\sqrt{36 - g_i^* - G_{-i}^*}} = 0$$

$$\Rightarrow \sqrt{36 - g_i^* - G_{-i}^*} = \frac{g_i^*}{2\sqrt{36 - g_i^* - G_{-i}^*}}$$

$$\Rightarrow 2(36 - g_i^* - G_{-i}^*) = g_i^*$$

$$\Rightarrow 72 - 2g_i^* - 2G_{-i}^* = g_i^*$$

$$\Rightarrow 72 - 2G_{-i}^* = 3g_i^*$$

$$\Rightarrow g_i^* = 24 - \frac{2}{3} G_{-i}^*$$

## Calculating the Nash Equilibrium

$$g_1^* = 24 - \frac{2}{3}(g_2^* + g_3^* + g_4^* + \ldots + g_n^*)$$
$$g_2^* = 24 - \frac{2}{3}(g_1^* + g_3^* + g_4^* + \ldots + g_n^*)$$
$$g_3^* = 24 - \frac{2}{3}(g_1^* + g_2^* + g_4^* + \ldots + g_n^*)$$
$$\vdots$$
$$g_n^* = 24 - \frac{2}{3}(g_1^* + g_2^* + g_3^* + \ldots + g_{n-1}^*)$$

Could use Linear Programming but notice the symmetry in these equations. It turns out that:

$g_1^* = g_2^* = \ldots = g_n^*$

If you don't believe me, try solving the 2 farmer case:

$$g_1^* = 24 - \frac{2}{3}g_2^*$$
$$g_2^* = 24 - \frac{2}{3}g_1^*$$

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## Calculating the Nash Equilibrium

$$g_1^* = 24 - \frac{2}{3}(g_2^* + g_3^* + g_4^* + \ldots + g_n^*)$$
$$g_2^* = 24 - \frac{2}{3}(g_1^* + g_3^* + g_4^* + \ldots + g_n^*)$$
$$g_3^* = 24 - \frac{2}{3}(g_1^* + g_2^* + g_4^* + \ldots + g_n^*)$$
$$\vdots$$
$$g_n^* = 24 - \frac{2}{3}(g_1^* + g_2^* + g_3^* + \ldots + g_{n-1}^*)$$

Write $g^* = g_1^* = g_2^* = \ldots = g_n^*$

$$g^* = 24 - \frac{2}{3}(n-1)g^*$$
$$=> 3g^* = 72 - 2(n-1)g^*$$
$$=> 3g^* + 2(n-1)g^* = 72$$
$$=> g^*(3 + 2n - 2) = 72$$
$$=> g^* = \frac{72}{2n+1}$$

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# Calculating the Nash Equilibrium

- At the Nash Equilibrium, a rational farmer grazes 72/(2n+1) goats
- How many goats in total will be grazed?

$$\frac{72n}{2n+1} = 36 - \frac{36}{2n+1}$$

- Note that as n $\rightarrow\infty$, 36 goats will be grazed (remember that we allow goats to be continuously divisible)

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# The Tragedy

- How much profit per farmer?

$$\text{Payoff to a farmer} = \frac{72}{2n+1}\sqrt{36 - \frac{72n}{2n+1}}$$

Suppose there are 24 farmers, then the payoff would be about 1.26 cents

If they all got together and agreed on 1 goat each, then the payoff would have been about 3.46 cents

$$\text{Payoff to a farmer} = \sqrt{36-24} = \sqrt{12} = 3.46$$

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## What Went Wrong?

- Rational behavior lead to sub-optimal solutions
- Maximizing one's utility is not the same as maximizing social welfare
- To solve this problem, we can define the rules of the game to ensure that social welfare is not disregarded
- This is why mechanism design is important since it involves defining the rules of the game



## Conclusions on Game Theory

- Sylvia Nasar's (author of the biography "A Beautiful Mind") synopsis of John Nash's remarks on winning the Nobel prize:

  "…he [Nash] felt that game theory was like string theory, a subject of great intrinsic intellectual interest that the world wishes to imagine can be of some utility. He said it with enough skepticism in his voice to make it funny."

# Conclusions on Game Theory

- Game theory is mathematically elegant but there are problems in applying it to real world problems:
  - Assumes opponents will play the equilibrium strategy
  - What to do with multiple Nash equilibria?
  - Computing Nash equilibria for complex games is nasty (perhaps even intractable)
  - Players have non-stationary policies
  - Lots of other assumptions that don't hold…
- Game theory used mainly to analyze environments at equilibrium rather than to control agents within an environment
- Also good for designing environments (mechanism design)



# What you should know

- How to calculate Nash Equilibria for a continuous action space game like the Tragedy of the Commons
- Why the Tragedy of the Commons is tragic
- Why game theory has difficulties being applied to real world problems