Q1. An experiment consists of tossing two six sided dice. Assume all outcomes have equal probability

(a) Find the sample space $S$.

(b) Find the probability of event $A$ that the sum of the dots on the dice equals 6.

(c) Find the probability of event $B$ that the sum of the dots on the dice is greater than 10.

(d) Find the probability of event $C$ that the sum of the dots on the dice is greater than 8 but less than 12.

Solution 1

(a) The sample space is

\[
\begin{align*}
&\{(1,1) \quad (1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (1,6)\} \\
&(2,1) \quad (2,2) \quad (2,3) \quad (2,4) \quad (2,5) \quad (2,6) \\
&(3,1) \quad (3,2) \quad (3,3) \quad (3,4) \quad (3,5) \quad (3,6) \\
&(4,1) \quad (4,2) \quad (4,3) \quad (4,4) \quad (4,5) \quad (4,6) \\
&(5,1) \quad (5,2) \quad (5,3) \quad (5,4) \quad (5,5) \quad (5,6) \\
&(6,1) \quad (6,2) \quad (6,3) \quad (6,4) \quad (6,5) \quad (6,6) \}
\end{align*}
\]

(b)

\[
A = \{(1,5) \quad (2,4) \quad (3,3) \quad (4,2) \quad (5,1)\}
\]

\[
P(A) = \frac{5}{36}
\]
(c) \[ B = \{(5, 6), (6, 5), (6, 6)\} \] 

\[ P(A) = \frac{3}{36} = \frac{1}{12} \]

(d) \[ C = \{(3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5)\} \]

\[ P(D) = \frac{9}{36} = \frac{1}{4} \]

Q2. In an experiment, \( A, B, C \) and \( D \) are events with probabilities 
\[ P[A] = \frac{1}{4}, \quad P[B] = \frac{1}{8}, \quad P[C] = \frac{5}{8}, \quad \text{and} \quad P[D] = \frac{3}{8}. \] 
Furthermore, \( A \) and \( B \) are disjoint, while \( C \) and \( D \) are independent.

(a) Find \( P[A \cap B], P[A \cup B], P[A \cap B^c], \) and \( P[A \cup B^c]. \)

(b) Are \( A \) and \( B \) independent?

(c) Find \( P[C \cap D], P[C \cap D^c], \) and \( P[C^c \cap D^c]. \)

(d) Are \( C^c \) and \( D^c \) independent?

Solution 2

(a) Since \( A \) and \( B \) are disjoint, \( P[A \cap B] = 0. \)
\[ P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{3}{8}. \]
It is obvious that \( A \subset B \) so that \( A \cap B = A. \) This implies 
\[ P[A \cap B^c] = P[A] = \frac{1}{4}. \]
It also follows that 
\[ P[A \cup B^c] = P[B^c] = 1 - \frac{3}{8} = \frac{7}{8}. \]

(b) Events \( A \) and \( B \) are not independent since 
\[ P[A \cap B] \neq P[A]P[B]. \]

(c) Since \( C \) and \( D \) are independent,
\[ P[C \cap D] = P[C]P[D] = \frac{15}{64}. \]
The next few items are a little trickier. We have 
\[ P[C \cap D^c] = P[C] - P[C \cap D] = \frac{5}{8} - \frac{15}{64} = \frac{25}{64}. \]
It follows that 
\[ P[C \cup D] = P[C] + P[D] - P[C \cap D] = \frac{5}{8} + (1 - 3/8) - \frac{25}{64} = \frac{55}{64}. \]
Using De Morgan’s law, we have 
\[ P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = \frac{15}{64}. \]
(d) Since $P[C \cap D] = P[C]P[D]$, $C$ and $D$ are independent.

Q3. Answer the following questions:


(b) Prove that $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

Solution 3

(a) It can be easily checked that the sets $A$ and $B \cap A$ are a partition of $A \cup B$. Then, $(A \cup B) = A \cup (B \cap \overline{A})$ implies that $P[A \cup B] = P[A] + P[B \cap \overline{A}]$. Similarly, set $B$ can be partitioned into sets $A \cap B$ and $B \cap A$: $B = (A \cap B) \cup (B \cap \overline{A})$ meaning that $P[B] = P[A \cap B] + P[B \cap \overline{A}]$. Therefore,


(b) In this question, we will be repeatedly using this axiom: if $X \cap Y = \emptyset$, then $P[X \cup Y] = P[X] + P[Y]$.

![Figure 1: The union of three sets, cited from Wikipedia.](image)

Let us consider the 7 disjoint subsets in Figure 1. The probability for each set is as follows:

1) $P[A] - P[A \cap C] - (P[A \cap B] - P[A \cap B \cap C])$
2) $P[B] - P[B \cap C] - (P[A \cap B] - P[A \cap B \cap C])$
3) $P[C] - P[B \cap C] - (P[A \cap C] - P[A \cap B \cap C])$
4) $P[B \cap C] - P[A \cap B \cap C]$
5) $P[A \cap B] - P[A \cap B \cap C]$
6) $P[A \cap C] - P[A \cap B \cap C]$
7) \( P[A \cap B \cap C] \)

By adding them together, the proof is completed.

Q4.

A number is selected uniformly at random from the set of integers \([-100, -99, \ldots, -1, 0, 1, \ldots, 99, 100]\)

What is the probability that it is divisible by 11, but neither by 3 nor by 5?

**Solution 4** The question can be simplified to What is the probability that it is divisible by 11, but neither by 3 nor by 5? Define three events as:

\[
A = \text{divisible by 3} \quad (8) \\
B = \text{divisible by 5} \quad (9) \\
C = \text{divisible by 11} \quad (10) \\
D = \text{divisible by 11, but not by 3 and 5} \quad (11)
\]

\[
P(C) = \frac{2 \times \lfloor 100/1 \rfloor + 1}{201} = \frac{19}{201} \quad (13) \\
P(C \cap A) = \frac{2 \times \lfloor 100/33 \rfloor + 1}{201} = \frac{7}{201} \quad (14) \\
P(C \cap B) = \frac{2 \times \lfloor 100/55 \rfloor + 1}{201} = \frac{3}{201} \quad (15) \\
P(C \cap A \cap B) = \frac{2 \times \lfloor 100/165 \rfloor + 1}{201} = \frac{1}{201} \quad (16)
\]

\[
P(D) = P(C) - P(C \cap A) - P(C \cap B) + P(C \cap A \cap B) = \frac{10}{201} \quad (17)
\]