1. Consider two random variables X and Y that follows the joint PDF:

\[ f_{XY}(x, y) = \begin{cases} 
  c, & x + y < 5, \ x \geq 0, \ y \geq 0, \\
  0, & \text{otherwise.}
\end{cases} \]  

(a) Find the value of \( c \).
(b) Prove that X and Y are not independent.

2. Let \( \{X(t) : t \geq 0\} \) be a Poisson process i.e., \( P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \). For \( s = t/5 \), show that the conditional distribution of \( X(s) \) given that \( X(t) = n \) is binomial with parameters \( n \) and \( p = 1/5 \), i.e.,

\[ P(X(t/5) = m | X(t) = n) = \binom{n}{m} (1 - p)^{n-m} p^m. \]

3. The joint PDF of X,Y is as follows.

\[ f_{XY}(x, y) = \begin{cases} 
  c e^{-x} e^{-y}, & x \geq 0, \ y \geq 0, \\
  0, & \text{otherwise.}
\end{cases} \]  

(a) Find the value of \( c \).
(b) Find \( f_Y(y) \).
(c) Find \( f_{X|Y}(x|y) \).
Homework 8 solution

Problem 1

1. We know that the joint PDF should satisfy
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x, y) \, dx \, dy = 1
\]
\[
\Rightarrow \int_{0}^{5} \int_{0}^{5-y} c \, dx \, dy = 1
\]
\[
\Rightarrow c \int_{0}^{5} (5-y) \, dy = 1
\]
\[
\Rightarrow c \left[ \frac{5y-y^2}{2} \right]_{0}^{5} = 1
\]
\[
\Rightarrow 25c/2 = 1
\]
\[
\Rightarrow c = 2/25.
\]

2. By definition of the marginal \( f_X(x) \) : \( f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dy \). Substituting \( f_{XY}(x, y) \) into the integral, we obtain
\[
f_X(x) = \int_{-\infty}^{\infty} 2/25dy
\]
\[
= 2/25 \int_{0}^{5-x} dy = 2/25(5-x), \quad 0 \leq x \leq 1.
\]

By definition of the marginal \( f_Y(y) \) : \( f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \). Substituting \( f_{XY}(x, y) \) into the integral, we obtain
\[
f_Y(y) = \int_{-\infty}^{\infty} 2/25dx = 2/25 \int_{0}^{5-y} dx = 2/25(5-y), \quad 0 \leq y \leq 1.
\]

Then, we have
\[
f_X(x)f_Y(y) = 2/25(5-x)2/25(5-y) = 4/625(5-x)(5-y) \neq f_{XY}(x, y).
\]

Therefore, \( X, Y \) are not independent.

Problem 2

\[
P(X(s) = m|X(t) = n) = P(X(t/5) = m|X(t) = n)
\]
\[
= \frac{P(X(t/5) = m, X(t) = n)}{P(X(t) = n)}
\]
\[
= \frac{P(X(t) = n|X(t/5) = m)P(X(t/5) = m)}{P(X(t) = n)}
\]
\[
= \frac{P(X(t) - X(t/5) = n-m)P(X(t/5) = m))}{P(X(t) = n)}
\]
\[
= \frac{P(X(t-t/5) = n-m)P(X(t/5) = m))}{P(X(t) = n)}
\]
As \(X(t)\) has Poisson distribution, \[P(X(s) = m|X(t) = n) = \frac{P(X(t - t/5) = n - m)P(X(t/5) = m)}{P(X(t) = n)}\]
\[= \frac{(\lambda(t-t/5))^{n-m}e^{-\lambda(t/5)}\{\lambda(t/5)\}^me^{-\lambda(t/5)}}{\frac{n!}{m!}\frac{n-m!}{m!}}\]
\[= \frac{n!}{(n-m)!m!}\frac{(\lambda(t-t/5))^{n-m}\{\lambda(t/5)\}^m e^{-\lambda(t/5)}e^{-\lambda(t/5)}}{e^{-\lambda(t)}}\]
\[= \frac{(n-m)!}{m!}\frac{t^{n-m}(1-1/5)^{n-m}(t^{m5-m})}{t^n}\]
\[= \frac{(n-m)!}{m!}(1-1/5)^{n-m}(1/5)^m\]
\[= \frac{(n-m)!}{m!}(1-p)^{n-m}p^m, \quad p = 1/5\]
So the conditional distribution of \(X(s)\) given that \(X(t) = n\) is binomial with parameters \(n\) and \(p = 1/5\).

**Problem 3**

a) \[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1
\]
\[
\Rightarrow \int_{0}^{\infty} \int_{0}^{\infty} ce^{-x}e^{-y} dy dx = 1,
\]
\[
\Rightarrow c \int_{0}^{\infty} e^{-x} \{\int_{0}^{\infty} e^{-y} dy\} dx = 1,
\]
\[
\Rightarrow c \int_{0}^{\infty} e^{-x}[1 - e^{-\infty}] dx = 1,
\]
\[
\Rightarrow c \int_{0}^{\infty} e^{-x} dx = 1,
\]
\[
\Rightarrow c e^{-x}|_{0}^{\infty} = 1,
\]
\[
\Rightarrow c = 1.
\]

b) We know \[f_{Y}(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx\]
\[= e^{-y} \int_{0}^{\infty} e^{-x} dx\]
\[= \begin{cases} 
  e^{-y}, & y \geq 0, \\
  0, & \text{o.w.} 
\end{cases}\]

c) By definition, the conditional PDF is as follows.
\[f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_{Y}(y)} \quad (5)\]
Substituting \(f_{XY}(x, y)\) and \(f_{Y}(y)\) in (5), we get
\[f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_{Y}(y)} = \frac{e^{-x}e^{-y}}{e^{-y}} = \begin{cases} 
  e^{-x}, & x \geq 0, \\
  0, & \text{o.w.} 
\end{cases}\]