Question 1 [11 marks]:
Consider a pn-junction as shown in Figure 1(a).

![Diagram of pn-junction](image)

**Figure 1** (a) Diagrammatic representation of pn-junction, with the semiconductors separated at \( x=0 \). (b) Electric field as a function of position in pn-junction. The depletion region spans: \( -w_p \leq x \leq w_n \).

The electric field is plotted as a function of position in Figure 1(b) and is given by Equation 1.

\[
E = -\frac{d\phi}{dx} = \begin{cases} 
\frac{N_A e}{\varepsilon_0 \varepsilon_r} (x + w_p) & -w_p < x < 0 \\
\frac{N_D e}{\varepsilon_0 \varepsilon_r} (x - w_n) & 0 < x < w_n \\
0 & \text{elsewhere}
\end{cases}
\]  

(1)

Where:
- \( E \) is electric field strength.
- \( \phi \) is electrostatic potential.
- \( x \) is position.
- \( N_A \) is acceptor density.
- \( N_D \) is the donor density.
- \( w_p \) is the width of the depletion region in the p-type semiconductor.
- \( w_n \) is the width of the depletion region in the n-type semiconductor.
- \( \varepsilon_0 \) is the vacuum permittivity.
- \( \varepsilon_r \) is the relative permittivity of the semiconductor.

If we define the electrostatic potential outside the p-type depletion region as zero, i.e. \( \phi(x = -w_p) = 0 \), then the electrostatic potential can be given by Equation 2.
\[
\phi = \begin{cases} 
\frac{N_A e}{2\varepsilon_0 \varepsilon_r} (x + w_p)^2 & -w_p < x < 0 \\
\Delta \phi_0 - \frac{eN_D}{2\varepsilon_0 \varepsilon_r} (x - w_n)^2 & 0 < x < w_n 
\end{cases}
\] (2)

Where \( \Delta \phi_0 \) is given by Equation 3:

\[
\Delta \phi_0 = \frac{e}{2\varepsilon_0 \varepsilon_r} \left( N_A w_p^2 + N_D w_n^2 \right)
\] (3)

The depletion widths are given by Equations (4) and (5):

\[
w_p = \left( \frac{2N_D \Delta \phi_0 \varepsilon_0 \varepsilon_r}{eN_A(N_D + N_A)} \right)^{1/2}
\] (4)

\[
w_n = \left( \frac{2N_A \varepsilon_0 \varepsilon_r \Delta \phi_0}{eN_D(N_D + N_A)} \right)^{1/2}
\] (5)

a) Integrate Equation 1 to show that the electrostatic potential \( \phi \) as a function of position \( x \) is given by Equation 2. Hint: since we define the electrostatic potential as \( \phi=0 \) at \( x = -w_p \), you should integrate the electric field from left-to-right across the whole junction, to the position you are interested in. [7 marks]

We are told that the integral of the electric field is the electrostatic potential:

\[ E = -\frac{d\phi}{dx} \]

Start with the p-type region \((-w_p < x < 0)\):

\[ E = -\frac{N_A e}{\varepsilon_0 \varepsilon_r} (x + w_p) \]

The main question here is setting the limits of the integral. We know we are only looking at the region: \(-w_p < x < 0\), however we are interested in the electrostatic at a position \( x \). So basically, for any position in the depletion region we must consider the electric field up to that point. Hence, we could say:

\[ \phi(x) = -\int_{-\infty}^{x} E(\tilde{x}) \, d\tilde{x} \]

Notice here we are using the dummy variable of integration \( \tilde{x} \). We know however from Equation 1 that there is zero electric field outside the depletion region:
\[
\int_{-\infty}^{-w_p} E(\bar{x}) \, d\bar{x} = 0
\]

So we can just say:

\[
\phi(x) = -\int_{-w_p}^{x} E(\bar{x}) \, d\bar{x}
\]

We can now carry out the integral:

\[
\phi(x) = -\int_{-w_p}^{x} E(\bar{x}) \, d\bar{x} = \int_{-w_p}^{x} \frac{N_A e}{\varepsilon_0 \varepsilon_r} (\bar{x} + w_p) \, d\bar{x}
\]

\[
\phi(x) = \frac{N_A e}{\varepsilon_0 \varepsilon_r} \left[ \frac{\bar{x}^2}{2} + \bar{x} w_p \right]_{-w_p}^{x}
\]

\[
\phi(x) = \frac{N_A e}{2 \varepsilon_0 \varepsilon_r} \left( \frac{x^2}{2} + xw_p - \frac{w_p^2}{2} + w_p^2 \right)
\]

\[
\phi(x) = \frac{N_A e}{2 \varepsilon_0 \varepsilon_r} \left( x^2 + 2xw_p + w_p^2 \right)
\]

We can then complete the square:

\[
\phi(x) = \frac{N_A e}{2 \varepsilon_0 \varepsilon_r} (x + w_p)^2
\]

As given above.

The electrostatic potential on the n-type side is harder to evaluate. Since we integrate the field all way from \(-\infty\) to \(x\), we can say the integral here is:

\[
\phi(x) = -\int_{-\infty}^{0} E(\bar{x}) \, d\bar{x} - \int_{0}^{x} E(\bar{x}) \, d\bar{x}
\]

Where the form of the electric field depends on where in the junction we are:

\[
\phi(x) = \int_{-\infty}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} (\bar{x} + w_p) \, d\bar{x} - \int_{0}^{x} \frac{N_D e}{\varepsilon_0 \varepsilon_r} (\bar{x} - w_n) \, d\bar{x}
\]

Let’s focus on the first integral. Again, we can move the lower limit on first integral from \(-\infty\) to \(-w_p\), since outside the depletion region the field strength is zero:

\[
\int_{-\infty}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} (\bar{x} + w_p) \, d\bar{x} = \int_{-w_p}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} (\bar{x} + w_p) \, d\bar{x}
\]
Notice this is a very similar integral to before, but this time we are integrating from $-w_p$ to 0 rather than from $-w_p$ to $x$:

\[
\int_{-w_p}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} (\bar{x} + w_p) \, d\bar{x} = \frac{N_A e \left( \bar{x}^2 + \bar{x} w_p \right)}{2 \varepsilon_0 \varepsilon_r}
\]

\[
\int_{-w_p}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} \left( \bar{x} + w_p \right) \, d\bar{x} = \frac{N_A e \left( 0 - 0 - \frac{w_p^2}{2} + w_p^2 \right)}{2 \varepsilon_0 \varepsilon_r}
\]

\[
\int_{-w_p}^{0} \frac{N_A e}{\varepsilon_0 \varepsilon_r} \left( \bar{x} + w_p \right) \, d\bar{x} = \frac{N_A e w_p^2}{2 \varepsilon_0 \varepsilon_r}
\]

Now consider the second integral:

\[
\int_{0}^{x} \frac{N_D e}{\varepsilon_0 \varepsilon_r} (\bar{x} - w_n) \, d\bar{x} = \frac{N_D e \left( \bar{x}^2 - \bar{x} w_n \right)}{2 \varepsilon_0 \varepsilon_r}
\]

\[
\int_{0}^{x} \frac{N_D e}{\varepsilon_0 \varepsilon_r} (\bar{x} - w_n) \, d\bar{x} = \frac{N_D e \left( \frac{x^2}{2} - x w_n - 0 + 0 \right)}{2 \varepsilon_0 \varepsilon_r}
\]

\[
\int_{0}^{x} \frac{N_D e}{\varepsilon_0 \varepsilon_r} (\bar{x} - w_n) \, d\bar{x} = \frac{N_D e (x^2 - 2x w_n)}{2 \varepsilon_0 \varepsilon_r}
\]

Putting the integrals together:

\[
\phi(x) = \frac{N_A e w_p^2}{2 \varepsilon_0 \varepsilon_r} - \frac{N_D e}{2 \varepsilon_0 \varepsilon_r} (x^2 - 2x w_n)
\]

\[
\phi(x) = \frac{e}{2 \varepsilon_0 \varepsilon_r} (2N_D x w_n + N_A w_p^2 - N_D x^2)
\]

We know we need to get out the factor $\Delta \phi_0$, where:

\[
\Delta \phi_0 = \frac{e}{2 \varepsilon_0 \varepsilon_r} (N_A w_p^2 + N_D w_n^2)
\]

To get this factor out we can add $N_D w_n^2 - N_D w_n^2 = 0$ to $\phi(x)$:

\[
\phi(x) = \frac{e}{2 \varepsilon_0 \varepsilon_r} (2N_D x w_n + N_A w_p^2 - N_D x^2 + N_D w_n^2 - N_D w_n^2)
\]

So we can write:
\[ \phi(x) = \Delta \phi_0 + \frac{e}{2\varepsilon_0 \varepsilon_r} (2N_D x w_n - N_D x^2 - N_D w_n^2) \]
\[ \phi(x) = \Delta \phi_0 - \frac{e N_D}{2\varepsilon_0 \varepsilon_r} (x^2 + w_n^2 - 2xw_n) \]

We can complete the square on the term in brackets:

\[ \phi(x) = \Delta \phi_0 - \frac{e N_D}{2\varepsilon_0 \varepsilon_r} (x - w_n)^2 \]

As per Equation 2.

b) By asserting that electric field is continuous at \( x = 0 \), use Equations 1, 2 and 3 to show that the width of the depletion regions (\( w_p \) and \( w_n \)) are given by Equations 4 and 5.[4 marks]

If the electric field is continuous at \( x = 0 \), all this means is that we can equate \( E(x = 0) \) for the two different regions:

\[ -\frac{N_A e}{\varepsilon_0 \varepsilon_r} (x + w_p) = \frac{N_D e}{\varepsilon_0 \varepsilon_r} (x - w_n) \]

This is true at \( x = 0 \) only:

\[ -\frac{N_A e w_p}{\varepsilon_0 \varepsilon_r} = -\frac{N_D e w_n}{\varepsilon_0 \varepsilon_r} \]

Hence:

\[ N_A w_p = N_D w_n \]

All we do it basically-rearrange Equation 3 so that it either has only one of \( w_p \) or \( w_n \). Let’s start with \( w_p \). First re-arrange \( N_A w_p = N_D w_n \):

\[ w_n = \frac{N_A w_p}{N_D} \]

Equation 3 is:

\[ \Delta \phi_0 = \frac{e}{2\varepsilon_0 \varepsilon_r} \left( N_A w_p^2 + N_D w_n^2 \right) \]

Substitute in \( w_n \):

\[ \Delta \phi_0 = \frac{e}{2\varepsilon_0 \varepsilon_r} \left( N_A w_p^2 + N_D \left( \frac{N_A w_p}{N_D} \right)^2 \right) \]
\[ \Delta \phi_0 = \frac{e}{2 \varepsilon_0 \varepsilon_r} \left( N_A w_p^2 + \frac{N_A^2 w_p^2}{N_D} \right) \]

Now, all we have to do is re-arrange it so \( w_p \) is the subject:

\[ \Delta \phi_0 = \frac{e w_p^2}{2 \varepsilon_0 \varepsilon_r} \left( N_A + \frac{N_A^2}{N_D} \right) \]

\[ \frac{2 \Delta \phi_0 \varepsilon_0 \varepsilon_r}{e} = w_p^2 \left( N_A + \frac{N_A^2}{N_D} \right) \]

\[ w_p^2 = \frac{2 \Delta \phi_0 \varepsilon_0 \varepsilon_r}{e \left( N_A + \frac{N_A^2}{N_D} \right)} \]

\[ w_p^2 = \frac{2 \Delta \phi_0 \varepsilon_0 \varepsilon_r}{e N_A \left( 1 + \frac{N_A}{N_D} \right)} \]

Multiple top and bottom by \( N_D \):

\[ w_p = \left( \frac{2 N_D \Delta \phi_0 \varepsilon_0 \varepsilon_r}{e N_A (N_D + N_A)} \right)^{1/2} \]

The process for \( w_n \) is very similar. First re-arrange \( N_A w_p = N_D w_n \) so that \( w_p \) is the subject:

\[ w_p = \frac{N_D w_n}{N_A} \]

Equation 3 is:

\[ \Delta \phi_0 = \frac{e}{2 \varepsilon_0 \varepsilon_r} \left( N_A w_p^2 + N_D w_n^2 \right) \]

Substitute in \( w_p \):

\[ \Delta \phi_0 = \frac{e}{2 \varepsilon_0 \varepsilon_r} \left( N_A \left( \frac{N_D w_n}{N_A} \right)^2 + N_D w_n^2 \right) \]

\[ \Delta \phi_0 = \frac{e}{2 \varepsilon_0 \varepsilon_r} \left( N_D^2 w_n^2 \frac{N_D}{N_A} + N_D w_n^2 \right) \]

\[ \Delta \phi_0 = \frac{e w_n^2}{2 \varepsilon_0 \varepsilon_r} \left( \frac{N_D^2}{N_A} + N_D \right) \]

\[ \frac{2 \varepsilon_0 \varepsilon_r \Delta \phi_0}{e} = w_n^2 \left( \frac{N_D^2}{N_A} + N_D \right) \]
\[ w_n^2 = \frac{2\varepsilon_0 \varepsilon_r \Delta \phi_0}{e \left( \frac{N_D^2}{N_A} + N_D \right)} \]

\[ w_n^2 = \frac{2\varepsilon_0 \varepsilon_r \Delta \phi_0}{e N_D \left( \frac{N_D}{N_A} + 1 \right)} \]

\[ w_n^2 = \frac{2N_A \varepsilon_0 \varepsilon_r \Delta \phi_0}{e N_D (N_D + N_A)} \]

\[ w_n = \left( \frac{2N_A \varepsilon_0 \varepsilon_r \Delta \phi_0}{e N_D (N_D + N_A)} \right)^{1/2} \]

**Question 2 [6 marks]:**

a) Figure 2 shows current-density-voltage characteristics \((J \text{ vs } V)\) of an example solar cell under illumination from \(P = 100 \text{ mW/cm}^2\) light. This \(J \text{ vs } V\) data is available to download using the link [here](#). Using this data, determine the efficiency of this solar cell. Give your answer in percent.[6 marks]

![Figure 2](image.png)

**Figure 2** current-density-voltage characteristics \((J \text{ vs } V)\) of an example solar cell under illumination from 100 mW/cm\(^2\) light. This data is available to download using the link [here](#).

First, let’s consider the equation for solar cell efficiency:

\[
\eta = \frac{V_{oc}J_{sc}FF}{P}
\]

We are given \(P = 100 \text{ mW/cm}^2\) in the question, but we need to evaluate the open-circuit voltage \((V_{oc})\), the short-circuit current \((J_{sc})\), and the fill-factor \((FF)\).

\(J_{sc}\) and \(V_{oc}\) are straight-forward to evaluate, as they are the current that flows at \(V = 0\), and the voltage required for zero current to flow, respectively. You can either evaluate them from the
graph, or extract them from the data. For example in Excel you could either scroll to where $V = 0$ and $J=0$, or use some lookup query. Either way, the values you get should be roughly:

\[ J_{sc} = 25 \text{ mA/cm}^2 \]

\[ V_{oc} = 1.09 \text{ V} \]

So now we have to evaluate the fill-factor ($FF$). Recall, the fill-factor is given by:

\[ FF = \frac{V_{mp}I_{mp}}{V_{oc}J_{sc}} = \frac{V_{mp}I_{mp}}{V_{oc}J_{sc}} \]

Where, $V_{mp}$ and $J_{mp}$ are the voltage and current, respectively, evaluated at the maximum power-point. We can already evaluate the denominator, but we do not know where the maximum power point lies, and hence we do not know $V_{mp}$ or $J_{mp}$. You can only evaluate the maximum power-point numerically, so here you are going to need to use the data provided.

There are several approaches one can take. For example you could evaluate power density as $P = VI$ for every point, then plot the data, and extract the values from the graph:

We could then use our approximated value of $V_{mp}$ to go back through our original data to find the corresponding current at this voltage ($J_{mp}$). Alternatively, you could use a function such as \texttt{INDIRECT("A"&MATCH(MAX(C2:C1092), C1:C1092, 0))} in Excel. Or you could use a numerical technique in a programing language. There are many options. You should end up with values roughly:

\[ J_{mp} = 23.2 \text{ mA/cm}^2 \]

\[ V_{mp} = 0.905 \text{ V} \]

With these numbers evaluated it is simply a matter of entering values. $FF$ is a ratio, so the units do not matter.

\[ FF = \frac{0.905 \times 23.2}{1.09 \times 25.0} \]
Now we can put this into our equation for efficiency to get the final answer:

\[ \eta = \frac{V_{oc} J_{sc} FF}{P} \]

\[ \eta = \frac{1.09 \times 25 \times 0.77}{100} \]

\[ \eta = 21\% \]

**Question 3 [8 marks]:**

a) When we use the word “oxide” in the context of metal-oxide semiconductor (MOS) capacitors, what type of electronic properties do we mean?[1 mark]

We use the word to describe insulators. While oxides can be conducting, semiconducting or insulating, the most common use oxide used in the microelectronics industry is silicon oxide (SiO₂). Hence why the word oxide is often used synonymously with insulator.

b) If we operate an n-type MOS capacitor in accumulation mode, what type of charge carriers would you expect to observe at the interface between the semiconductor and insulator?[1 mark]

Electrons.

c) If we operate an n-type MOS capacitor in depletion mode, what type of charge carriers would you expect to observe at the interface between the semiconductor and insulator?[1 mark]

None / ionized impurities.

d) If we operate an n-type MOS capacitor in inversion mode, what type of charge carriers would you expect to observe at the interface between the semiconductor and insulator? [1 mark]

Holes.

e) Figure 2 shows the capacitance per unit area of a MOS capacitor measured as a function of applied voltage (V) using a low frequency probe and a high frequency probe. From this data determine the capacitance the depletion region of this capacitor. Give your answer in \( \mu F/cm^2 \).[2 marks]
So we have two branches depending on the frequency at which we measure the capacitance. From Lecture 15 we know that in the low frequency regime (or in accumulation), the measured capacitance \( C_{LF} \) is just the capacitance of the oxide (insulator): \( C_i \):

\[
C_{LF} = C_i
\]

In the high frequency regime (or in depletion) the measured capacitance \( C_{HF} \) is due to the capacitance of the oxide \( (C_i) \) in series with the capacitance of the depletion region \( (C_D) \). To add the capacitance of two capacitors in series we say:

\[
\frac{1}{C_{HF}} = \frac{1}{C_D} + \frac{1}{C_i}
\]

Re-arranging:

\[
\frac{1}{C_D} = \frac{1}{C_{HF}} - \frac{1}{C_i}
\]

\[
\frac{1}{C_D} = \frac{C_i}{C_{HF}C_i} - \frac{C_{HF}}{C_{HF}C_i}
\]

\[
\frac{1}{C_D} = \frac{C_i - C_{HF}}{C_{HF}C_i}
\]

\[
C_D = \frac{C_{HF}C_i}{C_i - C_{HF}}
\]

We know from above that \( C_{LF} = C_i \):
\[ C_D = \frac{C_{HF} C_{LF}}{C_{LF} - C_{HF}} \]

We can read the numbers off of the graph (work in \( \mu F/cm^2 \)):

\[ C_D = \frac{2 \times 4}{4 - 2} = \frac{8}{2} = 4 \mu F/cm^2 \]

f) Say we are carrying out a capacitance measurement on a silicon-based MOS capacitor. For this capacitor, the thermal generation time is known to be 1 \( \mu \)s, the depletion width can be approximated to be 500 nm, and the capacitance per unit area of the oxide is 20 nF/cm\(^2\). The intrinsic carrier concentration of silicon is \( 10^{10} \) cm\(^3\). Determine the voltage ramp-rate required to observe inversion in this semiconductor. Give your answer in V/s.[2 marks]

In Lecture 6 we were given the inequality describing the rate of change of voltage required to observe inversion:

\[ \frac{dV}{dt} \leq \frac{e n_i W A}{\tau_g C_i} \]

We are given all the parameters here. We are given the capacitance as capacitance per unit area. Make sure to convert to SI units:

\[ \frac{C_i}{A} = 20 \text{ nF/cm}^2 = 20 \times 10^{-9} \text{ F/cm}^2 = 2 \times 10^{-4} \text{ F/m}^2 \]

Putting everything in in SI units:

\[ \frac{dV}{dt} \leq \frac{1.60 \times 10^{-19} \times 10^{16} \times 500 \times 10^{-9}}{1 \times 10^{-6} \times 2 \times 10^{-4}} \]

\[ \frac{dV}{dt} \leq 4 \text{ V/s} \]