# Some Solved Examples of Difference Equations

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# 8
# Worked Examples

We consider linear constant coefficient (LCC) $k^{\text{th}}$-order difference equations, that is, equations of the form

$$x_n - c_1 x_{n-1} - c_2 x_{n-2} \cdots - c_k x_{n-k} = \psi(n) \text{ for } n \geq k\,, \tag{8.1}$$

where $c_i \in \mathbb{C}$ with $c_k \neq 0$ and $\psi(x)$ is usually called the forcing function. Here we consider the special situation in which the forcing function $\psi(n)$ has the form

$$\psi(n) = \gamma^n \cdot p(n)$$

for $\gamma \in \mathbb{C}$ and certain polynomials $p(x) \in \mathbb{C}[x]$. We set $\lambda_1, \ldots, \lambda_t$ to be the distinct roots of the associated characteristic polynomial

$$ch(x) = x^k - c_1 x^{k-1} - \ldots - c_k\,.$$

## 8.1 Simple roots $(t = k)$

We know GIVE REF?

**Solving (8.1) when there are no multiple roots**

*When $ch(x)$ has no multiple roots, the equation (8.1) has the solution*

$$x_n = \sum_{i=1}^{k} a_i \lambda_i^n + \gamma^n \cdot q(n)\,, \tag{8.2}$$

*where $a_1, \dots, a_k \in \mathbb{C}$ and degree of the polynomial $q(x) \in \mathbb{C}[x]$ satisfies*

$$\deg(q) = \begin{cases} \deg(p) & \textit{if } \gamma \notin \{\lambda_1, \dots, \lambda_t\} \\ \deg(p) + 1 & \textit{if } \gamma \in \{\lambda_1, \dots, \lambda_t\} \end{cases}. \tag{8.3}$$

All examples in this section will be second-order LCC equations with characteristic polynomial

$$ch(x) = x^2 - x - 6 = (x-3)(x+2)$$

which has the simple roots $\lambda_1 = 3, \lambda_2 = -2$. Therefore, (8.2) and (8.3) become

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + \gamma^n \cdot q(n)\,, \tag{8.4}$$

where $a_1, a_2 \in \mathbb{C}$ and degree of $q(x)$ satisfies

$$\deg(q) = \begin{cases} \deg(p) & \text{if } \gamma \neq 3, -2 \\ \deg(p) + 1 & \text{if } \gamma = 3 \text{ or } \gamma = -2 \end{cases}. \tag{8.5}$$

*Example 8.1.1.* For the equation

$$x_n = x_{n-1} + 6x_{n-2},$$

$\psi(x)$ is the zero polynomial which gives $q(x) = 0$ as well; there exist $a_1, a_2 \in \mathbb{C}$ such that

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n. \tag{8.6}$$

For initial conditions $x_0, x_1$ we therefore have

$$x_0 = a_1 + a_2 \quad \text{and} \quad x_1 = 3a_1 - 2a_2. \tag{8.7}$$

Multiplying the first equation by 3 and subtracting that from the second equation, we obtain

$$x_1 - 3x_0 = -5a_2\ ;\ a_2 = \frac{3x_0 - x_1}{5}.$$

Inserting this value for $a_2$ into the first equation of (8.7) we obtain

$$a_1 = x_0 - \frac{3x_0 - x_1}{5} = \frac{2x_0 + x_1}{5};$$

the solution is

$$x_n = \frac{2x_0 + x_1}{5} 3^n + \frac{3x_0 - x_1}{5} (-2)^n.$$

How do we obtain a closed form for the equation

$$x_n = x_{n-1} + 6x_{n-2} + \psi(n)$$

when $\psi(x)$ is not the zero polynomial? It's often fairly simple to compute a particular solution. Once a particular solution $(v_n)$ is known then linearity allows us to use the general solution of the homogeneous equation to get the general solution

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + v_n.$$

This procedure is used in the next examples.

*Example 8.1.2.* Consider the second order equation

$$x_n = x_{n-1} + 6x_{n-2} + 2^n, \tag{8.8}$$

with $p(n) = 1$ and $\gamma = 2 \neq 3, -2$. Then the general solution is

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + q(n) \cdot 2^n \,,$$

for $a_1, a_2 \in \mathbb{C}$ and $q(x) \in \mathbb{C}[x]$ with $\deg(q) = \deg(p) = 0$; $q(x) = c \in \mathbb{C}$ and we therefore obtain

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n + c \cdot 2^n$$

for $a_1, a_2, c \in \mathbb{C}$. For $a_1 = a_2 = 0$ we obtain the solution $v_n = c \cdot 2^n$, where

$$v_{n-1} + 6v_{n-2} + 2^n = c2^{n-1} + 6c2^{n-2} + 2^n = 2^{n-1}(c + 3c + 2).$$

Since $v_n$ is the general term of a solution to (8.8),

$$c \cdot 2^n = v_n = 2^{n-1}(4c + 2) \;;\; 2c = 4c + 2 \;;\; c = -1 \;;$$

that is, $v_n = -2^n$ and

$$x_n = a_1 \cdot 3^n + a_2 \cdot (-2)^n - 2^n, \text{ for some } a_1, a_2 \in \mathbb{C}. \tag{8.9}$$

For initial values $x_0, x_1$ we therefore obtain

$$x_0 = a_1 + a_2 - 1 \text{ and } x_1 = 3a_1 - 2a_2 - 2, \tag{8.10}$$

which we want to solve for $a_1, a_2$ as in the previous example. Multiplying the first equation by 2 and adding it to the second, we have

$$2x_0 + x_1 = 5a_1 - 4 \ ; \ a_1 = \frac{2x_0 + x_1 + 4}{5}.$$

Substituting this value of $a_1$ into the first equation of (8.10) we obtain

$$a_2 = x_0 - a_1 + 1 = x_0 - \frac{2x_0 + x_1 + 4}{5} + 1 = \frac{3x_0 - x_1 + 1}{5},$$

and (8.9) becomes

$$x_n = \frac{2x_0 + x_1 + 4}{5} \cdot 3^n + \frac{3x_0 - x_1 + 1}{5} \cdot (-2)^n - 2^n.$$

Checking the next term of the sequence we have

$$x_2 = \frac{2x_0 + x_1 + 4}{5} \cdot 9 + \frac{3x_0 - x_1 + 1}{5} \cdot 4 - 4 = x_1 + 6x_0 + 4,$$

as required by the recurrence.

*Example 8.1.3.* The second-order equation

$$x_n = x_{n-1} + 6x_{n-2} + n2^n \tag{8.11}$$

has $p(n) = n$ (with $\deg(p) = 1$) and $\gamma = 2 \neq 3, -2$; $v_n = (an + b) \cdot 2^n$ for constants $a, b$ and

$$\begin{aligned} v_{n-1} + 6v_{n-2} + n2^n - v_n & \\ &= (an - a + b)2^{n-1} + 6(an - 2a + b) \cdot 2^{n-2} + n2^n - (an + b)2^n \\ &= 2^{n-1}((an - a + b) + 3(an - 2a + b) + 2n - 2(an + b)) \\ &= 2^{n-1}((2a + 2)n + (-7a + 2b)), \end{aligned}$$

which equals zero for all $n \geq 0$ when $2a + 2 = 0$ and $-7a + b = 0$; $a = -1$ and $2b = 7a = -7$. Therefore, $v_n = -(n + \frac{7}{2})2^n$ is a particular solution of (8.11) and the general solution has the form

$$x_n = a_1 3^n + a_2(-2)^n - (n + \frac{7}{2})2^n \ \text{ for } a_1, a_2 \in \mathbb{C}.$$

For any initial values $x_0, x_1$ from this we obtain:

$$x_0 = a_1 + a_2 - \frac{7}{2} \ \text{ and } x_1 = 3a_1 - 2a_2 - 9.$$

Simultaneously solving this system of equations, we obtain

$$a_1 = \frac{2x_0 + x_1 + 16}{5} \ \text{ and } \ a_2 = \frac{3x_0 - x_1 + \frac{3}{2}}{5};$$

$$x_n = \frac{2x_0 + x_1 + 16}{5} \cdot 3^n + \frac{3x_0 - x_1 + \frac{3}{2}}{5} \cdot (-2)^n - \Big(n + \frac{7}{2}\Big)2^n.$$

Verifying this calculation for $n = 2$:

$$x_2 = \frac{18x_0 + 9x_1 + 144}{5} + \frac{12x_0 - 4x_1 + 6}{5} - \Big(2 + \frac{7}{2}\Big)4 \quad = 6x_0 + x_1 + 8,$$

as required.

*Example 8.1.4.* For

$$x_n = x_{n-1} + 6x_{n-2} + 3^n,$$

$p(x) = 1$ and $\gamma = 3 = \lambda_1$, from which we obtain

$$x_n = a_1 3^n + a_2(-2)^n + 3^n \cdot q(n)$$

for polynomial $q(x)$ with $\deg(q) = 1$. Therefore,

$$x_n = a_1 3^n + a_2(-2)^n + 3^n(a + bn) = (a_1 + a)3^n + a_2(-2)^n + bn3^n \text{ .,}$$

and we can choose $v_n = bn \cdot 3^n$ for $b \in \mathbb{C}$. (Note how the constant term of $q(x)$ has been absorbed into the earlier coefficient of $3^n$.) Then

$$v_{n-1} + 6v_{n-2} + 3^n - v_n = 3^{n-1}b(n - 1 + 2n - 4 - 3n) + 3^n = 3^{n-1}(-5b + 3);$$

from this, $b = \frac{3}{5}$ and $v_n = \frac{1}{5}n3^{n+1}$. Therefore,

$$x_0 = a_1 + a_2 \quad \text{and} \quad x_1 = 3a_1 - 2a_2 + \frac{9}{5},$$

which gives

$$a_1 = \frac{2x_0 + x_1 - \frac{9}{5}}{5} \ ; \ a_2 = \frac{3x_0 - x_1 + \frac{9}{5}}{5} \, .$$

Hence,

$$x_n = \frac{1}{5}\Big((2x_0 + x_1 - \frac{9}{5})3^n + (3x_0 - x_1 + \frac{9}{5})(-2)^n + n3^{n+1}\Big).$$

*Example 8.1.5.* For

$$x_n = x_{n-1} + 6x_{n-2} + n(-2)^n$$

$p(x) = 1$ and $\gamma = -2 = \lambda_2$, from which we obtain

$$x_n = a_1 3^n + a_2(-2)^n + (-2)^n \cdot q(n)$$

for $q(x)$ with $\deg(q) = 2$. Therefore, again absorbing the constant term of $q(x)$, there exist $b, c \in \mathbb{C}$ such that

$$x_n = a_1 3^n + a_2(-2)^n + (-2)^n(bn + cn^2)\,;$$

$$x_n = a_1 3^n + a_2(-2)^n + v_n, \tag{8.12}$$

where $v_n = (cn^2 + bn)(-2)^n$. Then

$$\begin{aligned} &v_{n-1} + 6v_{n-2} + (-2)^n - v_n \\ &\qquad =(-2)^{n-1}(c(n-1)^2 + b(n-1) - 3(c(n-2)^2 + b(n-2)) - 2n + 2(cn^2 + bn)) \\ &\qquad =(-2)^{n-1}((10c-2)n + (5b - 11c)) \end{aligned}$$

equals zero for all $n \geq 0$ when

$$10c = 2 \text{ and } 5b = 11c\,;$$

that is, $c = \frac{1}{5}, b = \frac{11}{25}$ and $v_n = (\frac{11}{25} + \frac{1}{5}n)n \cdot (-2)^n$. Therefore, putting $v_0 = 0$ and $v_1 = \frac{32}{5}$ in (8.12)we obtain

$$x_0 = a_1 + a_2 \quad \text{and} \quad x_1 = 3a_1 - 2a_2 - \frac{32}{5}$$

and from this

$$a_1 = \frac{2x_0 + x_1 + \frac{32}{5}}{5}\ ;\ a_2 = \frac{3x_0 - x_1 - \frac{32}{5}}{5}\,.$$

Hence,

$$x_n = \frac{1}{5}\Big(\Big(2x_0 + x_1 + \frac{32}{5}\Big)3^n + \Big(3x_0 - x_1 - \frac{32}{5} + \frac{11}{25}n + \frac{1}{5}n^2\Big)(-2)^n\Big).$$

## 8.2 Multiple roots ($t < k$)

INSERT THE REFERENCE FOR THE NEXT:

**Solution to (8.1) when there is one multiple root**

*When $ch(x)$ has $\lambda_1 = \lambda_2$ and all of $\lambda_3, \ldots, \lambda_k$ are distinct,the equation (8.1) has the solution*

$$x_n = (a_1 n + a_2)\lambda_1^n + \sum_{i=3}^{k} a_i \lambda_i^n + \gamma^n \cdot q(n)\,, \tag{8.13}$$

*where $a_1, \ldots, a_k \in \mathbb{C}$ and the degree of the polynomial $q(x) \in \mathbb{C}[x]$ satisfies*

$$\deg(q) = \begin{cases} \deg(p) & \text{if } \gamma \notin \{\lambda_1, \ldots, \lambda_t\} \\ \deg(p) + 1 & \text{if } \gamma \in \{\lambda_3, \ldots, \lambda_t\} \\ \deg(p) + 2 & \text{if } \gamma = \lambda_1 \end{cases}. \tag{8.14}$$

The first five examples in this section are second-order linear constant coefficient recurrence equations with characteristic polynomial

$$ch(x) = x^2 - 4x + 4 = (x-2)^2,$$

which has the double root $\lambda_1 = \lambda_2 = 2$. Therefore, (8.13) and (8.14) become

$$x_n = (a_1 n + a_2)2^n + \gamma^n \cdot q(n)\,, \tag{8.15}$$

where $a_1, a_2 \in \mathbb{C}$ and degree of the polynomial $q(x) \in \mathbb{C}[x]$ satisfies

$$\deg(q) = \begin{cases} \deg(p) & \text{if } \gamma \neq 2 \\ \deg(p) + 2 & \text{if } \gamma = 2 \end{cases}. \tag{8.16}$$

*Example 8.2.1.* For the equation

$$x_n = 4x_{n-1} - 4x_{n-2},$$

$\psi(x)$ is the zero polynomial which gives $q(x) = 0$; there exist $a_1, a_2 \in \mathbb{C}$ such that

$$x_n = (a_1 n + a_2)2^n. \tag{8.17}$$

For initial conditions $x_0, x_1$ we can compute $a_2 = x_0$ and $a_1 = \dfrac{x_1 - 2x_0}{2}$, which gives the general term

$$x_n = ((x_1 - 2x_0)n + 2x_0)2^{n-1}.$$

As we observed in the previous section when we considered the case of simple roots, if $(v_n)$ is any particular solution to the given equation then the general solution has the form

$$x_n = (a_1 n + a_2)2^n + v_n,$$

for some $a_1, a_2 \in \mathbb{C}$.

*Example 8.2.2.* Consider the second order equation

$$x_n = 4x_{n-1} - 4x_{n-2} + 3^n, \tag{8.18}$$

with $p(n) = 1$ and $\gamma = 3 \neq 2$. Then the general solution is

$$x_n = (a_1 n + a_2)2^n + q(n) \cdot 3^n,$$

for $a_1, a_2 \in \mathbb{C}$ and $q(x) \in \mathbb{C}[x]$ with $\deg(q) = \deg(p) = 0$; $q(x) = c \in \mathbb{C}$ and

$$x_n = (a_1 n + a_2)2^n + c3^n.$$

For $a_1 = a_2 = 0$ we obtain the solution $v_n = c \cdot 3^n$, where

$$4v_{n-1} - 4v_{n-2} + 3^n - v_n = 3^{n-2}(12c - 4c + 9 - 9c) = 3^{n-2}(9 - c),$$

which equals zero for $c = 9$. From this we obtain the particular solution with general term $v_n = 3^{n+2}$ and

$$x_n = (a_1 n + a_2)2^n + 3^{n+2}, \text{ for some } a_1, a_2 \in \mathbb{C}. \tag{8.19}$$

For initial conditions $x_0, x_1$ computation gives

$$x_n = \Big(\frac{x_1 - 2x_0 - 9}{2} n + (x_0 - 9)\Big)2^n + 3^{n+2}.$$

*Example 8.2.3.* The second-order equation

$$x_n = 4x_{n-1} - 4x_{n-2} + n3^n \tag{8.20}$$

has $\deg(p) = 1$ and $\gamma = 3 \neq 2$. Therefore, we use $v_n = (an + b) \cdot 3^n$ for constants $a, b$ and have

$$\begin{aligned} 4v_{n-1} &- 4v_{n-2} + n3^n - v_n \\ &= 3^{n-2}(12(an - a + b) - 4(an - 2a + b) + 9n - 9(an + b)) \\ &= 3^{n-2}((9 - a)n - (4a + b)), \end{aligned}$$

which equals zero when $a = 9$ and $b = -4a = -36$. Therefore, $v_n = (n - 4)3^{n+2}$ is a particular solution of (8.20) and the general solution has the form

$$x_n = (a_1 n + a_2)2^n + (n - 4)3^{n+2}.$$

For initial values $x_0, x_1$ we obtain:

$$x_n = \Big(\frac{x_1 - 2x_0 + 9}{2} n + (x_0 + 36)\Big)2^n + (n - 4)3^{n+2}.$$

*Example 8.2.4.* The second-order equation

$$x_n = 4x_{n-1} - 4x_{n-2} + 2^n \tag{8.21}$$

has $\deg(p) = 0$ and $\gamma = 2 = \lambda_1$. Therefore, $\deg(p) = 2$ and let $v_n = an^2 2^n$ for $a \in \mathbb{C}$. Then

$$\begin{aligned}
4v_{n-1} - 4v_{n-2} + 2^n - v_n &\\
&= 2^n(2a(n-1)^2 - a(n-2)^2 + 1 - an^2)\\
&= 2^n(-2a+1),
\end{aligned}$$

implying for $a = \frac{1}{2}$, the sequence with general term $v_n = n^2 2^{n-1}$ is a particular solution of (8.21). The general solution has the form

$$x_n = (n^2 + a_1 n + a_2)2^{n-1}.$$

For initial values $x_0, x_1$ we obtain:

$$x_n = (n^2 + (x_1 - 1 - 2x_0)n + 2x_0)2^{n-1}.$$

*Example 8.2.5.* The second-order equation

$$x_n = 4x_{n-1} - 4x_{n-2} + n2^n \tag{8.22}$$

has $\deg(p) = 1$ and $\gamma = 2 = \lambda_1$. Therefore, $\deg(p) = 3$; $v_n = (an^3 + bn^2)2^n$ for $a, b \in \mathbb{C}$. Then

$$\begin{aligned}
&4v_{n-1} - 4v_{n-2} + n2^n - v_n\\
&\quad = 2^n(2(a(n-1)^3 + b(n-1)^2) - (a(n-2)^3 + b(n-2)^2) + n - (an^3 + bn^2))\\
&\quad = 2^n[-6an - 2b + 6a + n] = 2^n[n(1-6a) + 2(3a-b)],
\end{aligned}$$

which equals zero when $a = \frac{1}{6}$ and $b = 3a = \frac{1}{2}$; $v_n = (\frac{1}{3}n^3 + n^2)2^{n-1}$ is a particular solution of (8.22). The general solution therefore has the form

$$x_n = (\frac{1}{3}n^3 + n^2 + a_1 n + a_2)2^{n-1}.$$

Initial values $x_0, x_1$ give

$$x_n = ((\frac{1}{3}n^3 + n^2 + (x_1 - 2x_0 - \frac{4}{3})n + 2x_0)2^{n-1}.$$

Since each of the last five examples had only one (double) root, it was impossible to be in the middle alternative of (8.14). The next example illustrates that case.

*Example 8.2.6.* The second-order equation

$$x_n = 4x_{n-1} - 5x_{n-2} + 2x_{n-3} + 2^n \tag{8.23}$$

has $\deg(p) = 0$ and $\gamma = 2$. Since the characteristic polynomial is

$$ch(x) = x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2),$$

$\gamma$ is a simple root of $ch(x)$ and the general solution is

$$x_n = (a_1 n + a_2) + a_3 2^n + 2^n \cdot q(n)\,,$$

where $a_1, a_2, a_3 \in \mathbb{C}$ and $\deg(q) = 1$. Setting $v_n = an2^n$,

$$\begin{aligned}&4v_{n-1} - 5v_{n-2} + 2v_{n-3} + 2^n - v_n\\ &\qquad = 2^{n-2}(8a(n-1) - 5a(n-2) + a(n-3) + 4 - 4an) = 2^{n-2}(4-a);\end{aligned}$$

$v_n = 4n2^n = n2^{n+2}$ is a particular solution of (8.23), and the general solution has

$$x_n = (a_1 n + a_2) + a_3 2^n + n2^{n+2}.$$

For initial values $x_0, x_1, x_2$ we obtain

$$x_0 = a_2 + a_3\,;\; x_1 = a_1 + a_2 + 2a_3 + 8\,;\; x_2 = 2a_1 + a_2 + 4a_3 + 32\,,$$

which implies

$$\begin{aligned}x_n = (-2x_0 + 3x_1 - x_2 + 8)n + (2x_1 - x_2 + 16)\\ +(x_0 - 2x_1 + x_2 - 16)2^n + n2^{n+2}.\end{aligned}$$