Difference Equations

Solve $f_n = f_{n-1} + f_{n-2}$ $\quad\quad f_0 = 2, f_1 = 1$

Step 1: Find Roots

$$f_n = f_{n-1} + f_{n-2}$$

$$f_n - f_{n-1} - f_{n-2} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$(1 \pm \sqrt{(1^2 - 4*1*(-1))})/(2*1) = (1 \pm \sqrt{5})/2$$

$$\Rightarrow \lambda_1 = (1+\sqrt{5})/2, \quad \lambda_2 = (1-\sqrt{5})/2$$

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Step 2: Find General Solution

$$f_n = a_1\lambda_1^n + a_2 \lambda_2^n$$

substitute in initial conditions and solve

$$n = 0: \quad f_0 = 2 = a_1\lambda_1^0 + a_2 \lambda_2^0 = a_1 + a_2$$

$$2 - a_1 = a_2$$

$$n = 1: \quad f_1 = 1 = a_1\lambda_1^1 + a_2 \lambda_2^1$$

$$1 = a_1\lambda_1 + (2 - a_1) \lambda_2$$

$$1 = a_1\lambda_1 + 2 \lambda_2 - a_1\lambda_2$$

$$1 - 2 \lambda_2 = (\lambda_1 - \lambda_2)\, a_1$$

$$a_1 = (1 - 2 \lambda_2)/ (\lambda_1 - \lambda_2)$$

$$a_1 = 1 \quad \Rightarrow a_2 = 2 - 1 = 1$$

$$\Rightarrow f_n = 1*\lambda_1^n + 1*\lambda_2^n = ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n$$

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Solve $x_n = x_{n-1} + 2x_{n-2}$ $x_0 = 1, x_1 = 1$

Step 1: Find Roots

$x_n = x_{n-1} + 2x_{n-2}$

$x_n - x_{n-1} - 2x_{n-2} = 0$

$\lambda^2 - \lambda - 2 = 0$

$(1 \pm \sqrt{}(1^2 - 4*1*(-2)))/(2*1) = (1 \pm \sqrt{}9)/2$

$\Rightarrow \lambda_1 = (1+\sqrt{}9)/2 = 2, \lambda_2 = (1-\sqrt{}9)/2 = -1$

Step 2: Find General Solution

$x_n = a_1 \lambda_1^n + a_2 \lambda_2^n$

substitute in initial conditions and solve

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n = 0: $x_0 = 1 = a_1 \lambda_1^0 + a_2 \lambda_2^0$

$1 = a_1 + a_2$

$1 - a_1 = a_2$

n = 1: $x_1 = 1 = a_1 \lambda_1^1 + a_2 \lambda_2^1$

$1 = a_1 \lambda_1 + (1 - a_1)\lambda_2$

$1 = a_1 \lambda_1 + \lambda_2 - a_1\lambda_2$

$1 - \lambda_2 = (\lambda_1 - \lambda_2)\, a_1$

$a_1 = (1 - \lambda_2)/ (\lambda_1 - \lambda_2)$

$a_1 = (1 - (-1))/(2 - (-1)) = 2/3$

$\Rightarrow a_2 = 1 - 2/3 = 1/3$

$\Rightarrow x_n = 2/3\, \lambda_1^n + 1/3\, \lambda_2^n = 2/3(2)^n + 1/3(-1)^n$

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Solve $x_n = 2x_{n-1} + 3x_{n-2}$ $x_0 = 1, x_1 = 1$

Step 1: Find Roots

$x_n = 2x_{n-1} + 3x_{n-2}$

$x_n - 2x_{n-1} - 3x_{n-2} = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(2 \pm \sqrt{(2^2 - 4*1*(-3))})/(2*1) = (2 \pm \sqrt{(16)})/2$

$\Rightarrow \lambda_1 = (2 + \sqrt{(16)})/2 = 3$

$\lambda_2 = (2 - \sqrt{(16)})/2 = -1$

Step 2: Find General Solution

$x_n = a_1\lambda_1^n + a_2\lambda_2^n$

substitute in initial conditions

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n = 0: $x_0 = 1 = a_1\lambda_1^0 + a_2\lambda_2^0$

$1 = a_1 + a_2$

$1 - a_1 = a_2$

n = 1: $x_1 = 1 = a_1\lambda_1^1 + a_2\lambda_2^1$

$1 = 3a_1 + (-1)(1 - a_1)$

$1 = 3a_1 - 1 + a_1$

$2 = 4a_1$

$a_1 = 1/2 \quad \Rightarrow a_2 = 1 - 1/2 = 1/2$

$\Rightarrow x_n = 1/2(3^n + (-1)^n)$

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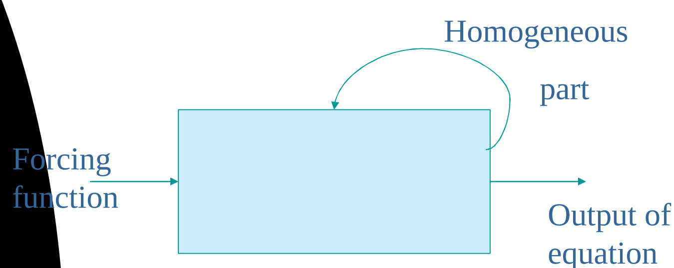
Forcing Functions

Equations of the form:

$$x_n = \sum_{i=1}^{n} c_i x_{n-i} + g(n)$$

where g(n) is the forcing function



Output will look like either the homogeneous solution or the forcing function, which ever is larger

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Solve: $x_n = 3x_{n-1} + 5$ $\quad x_0 = 1$

$x_n = h_n + v_n$

$h_n = 3h_{n-1}$

$v_n = 3v_{n-1} + 5$

Step 1: Find Roots

$h_n = 3h_{n-1}$

$h_n - 3h_{n-1} = 0$

$\lambda - 3 = 0$

$\lambda = 3$

$\Rightarrow h_n = a_1 3^n$

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Step 2: Find Particular Solution

$v_n = 3v_{n-1} + 5$

Guess v_n = general equation of same form as forcing function

GUESS: $v_n = c$ where c = constant

Solve for variables by substituting in guess

$v_n = 3v_{n-1} + 5$

$c = 3c + 5$

$-2c = 5$

$c = -5/2$

Substitute results into the guess equation

$\Rightarrow v_n = -5/2$

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Step 3: Find General Solution

$h_n = a_1 3^n$

$x_n = h_n + v_n$

$x_n = a_1 3^n - 5/2$

substitute in initial conditions and solve

n = 0: $x_0 = 1 = a_1 3^0 - 5/2$

$1 = a_1 - 5/2$

$7/2 = a_1$

$\Rightarrow x_n = 7/2 * 3^n - 5/2$

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Solve $x_n = 3x_{n-1} + 2n - 4$ $x_0 = 1$

$x_n = h_n + v_n$
$h_n = 3h_{n-1}$
$v_n = 3v_{n-1} + 2n - 4$

Step 1: Find Roots

$h_n = 3h_{n-1}$

$h_n - 3h_{n-1} = 0$

$\lambda - 3 = 0$

$\lambda = 3$

$\Rightarrow h_n = a_1 3^n$

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Step 2: Find Particular Solution

$v_n = 3v_{n-1} + 2n - 4$

Guess v_n = general equation of same form as forcing function

GUESS: $v_n = c_1 n + c_0$

Solve for variables by substituting in guess

$v_n = 3v_{n-1} + 2n - 4$

$c_1 n + c_0 = 3(c_1(n-1) + c_0) + 2n - 4$

$c_1 n + c_0 = 3c_1 n - 3c_1 + 3c_0 + 2n - 4$

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Equation from last slide: $c_1 n + c_0 = 3c_1 n - 3c_1 + 3c_0 + 2n - 4$

pull out terms by their power

n: $c_1 n = 3c_1 n + 2n$

$c_1 = 3c_1 + 2$

$-2 = 2 c_1$

$c_1 = -1$

1: $c_0 = - 3c_1 + 3c_0 - 4$

$c_0 = - 3(-1) + 3c_0 - 4$ //sub in answer

$c_0 = 3 + 3c_0 - 4$

$-2 c_0 = - 1$

$c_0 = 1/2$

$\Rightarrow v_n = c_1 n + c_0 = -1n + 1/2 = -n + 1/2$

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Why Can Break Up By Powers

if $ax^3 + bx^2 + cx + d = ex^3 + fx^2 + gx + h$

this means that they = 0 for the same values of n.

Therefore, $0 = ax^3 + bx^2 + cx + d$

$0 = ex^3 + fx^2 + gx + h$

this means that a = e, b = f, c = g, and d = h otherwise the polynomials wouldn't be equal

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Step 3: Find General Solution

$h_n = a_1 3^n$

$x_n = h_n + v_n$

$x_n = a_1 3^n - n + 1/2$

substitute in initial conditions and solve

n = 0: $x_0 = 1 = a_1 3^0 - 0 + 1/2$

$1 = a_1 + 1/2$

$1/2 = a_1$

$\Rightarrow x_n = 1/2 * 3^n - n + 1/2$

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Solve $x_n = 3x_{n-1} + 2^n$ $x_0 = 1$

$x_n = h_n + v_n$
$h_n = 3h_{n-1}$
$v_n = 3v_{n-1} + 2^n$

Step 1: Find Roots

$h_n = 3h_{n-1}$

$h_n - 3h_{n-1} = 0$

$\lambda - 3 = 0$

$\lambda = 3$

$\Rightarrow h_n = a_1 3^n$

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Step 2: Find Particular Solution

$v_n = 3v_{n-1} + 2^n$

Guess v_n = general equation of same form as forcing function

GUESS: $v_n = c\, 2^n$

Solve for variables by substituting in guess

$v_n = 3v_{n-1} + 2^n$

$c\, 2^n = 3(c\, 2^{n-1}) + 2^n$

$c\, 2 = 3c + 2$ //divide both sides by 2^{n-1}

$-c = 2$

$c = -2$

$\Rightarrow v_n = -2*2^n = -(2^{n+1})$

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Step 3: Find General Solution

$h_n = a_1 3^n$

$x_n = h_n + v_n$

$x_n = a_1 3^n - (2^{n+1})$

substitute in initial conditions and solve

$n = 0$: $x_0 = 1 = a_1 3^0 - (2^{0+1})$

$1 = a_1 - 2$

$3 = a_1$

$\Rightarrow x_n = 3 * 3^n - 2^{n+1} = 3^{n+1} - 2^{n+1}$

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