Problem 2.32

From

\[\delta[n-k] * h[n] = h[n-k]\]
\[(ax_1[n] + bx_2[n]) * h[n] = ax_1[n] * h[n] + bx_2[n] * h[n]\]

given that \[h[n] = \delta[n+1] + 3\delta[n] + 2\delta[n-1] - \delta[n-2] + \delta[n-3]\]

(a) \[x[n] = 3\delta[n] - 2\delta[n-1]\]

\[y[n] = x[n] * h[n] = \{3\delta[n] - 2\delta[n-1]\} * h[n] = 3h[n] - 2h[n-1] = 3\delta[n+1] + 9\delta[n] + 6\delta[n-1] - 3\delta[n-2] + 3\delta[n-3] - 2\delta[n-4] - 6\delta[n-1] - 4\delta[n-2] + 2\delta[n-3] - 2\delta[n-4]\]

(b) \[x[n] = u[n+1] - u[n-3] = \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\]

\[y[n] = x[n] * h[n] = \{\delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\} * h[n] = h[n+1] + h[n] + h[n-1] + h[n-2] = \delta[n+2] + 4\delta[n+1] + 6\delta[n] + 5\delta[n-1] + 5\delta[n-2] + 2\delta[n-3] + \delta[n-5]\]

(c) \[x[n]\] as given in Fig. P2.32 (b)

\[x[n] = 2\delta[n-3] + 2\delta[n] - \delta[n+2]\]
\[y[n] = 2h[n-3] + 2h[n] - h[n+2] = -\delta[n+3] - 3\delta[n+2] + 7\delta[n] + 3\delta[n-1] + 8\delta[n-3] + 4\delta[n-4] - 2\delta[n-5] + 2\delta[n-6]\]

2.38. An LTI system has impulse response \(h(t)\) depicted in Fig. P2.38. Use linearity and time invariance to determine the system output \(y(t)\) if the input \(x(t)\) is

(a) \(x(t) = 2\delta(t+2) + \delta(t-2)\)

\[y(t) = 2h(t+2) + h(t-2)\]

(b) \(x(t) = \delta(t-1) + \delta(t-2) + \delta(t-3)\)

\[y(t) = h(t-1) + h(t-2) + h(t-3)\]

(c) \(x(t) = \sum_{p=0}^{\infty} (-1)^p \delta(t-2p)\)

\[y(t) = \sum_{p=0}^{\infty} (-1)^p h(t-2p)\]
The plot of the $y(t)$ for part (a), the time-shifting version of original $h(t)$ with some multiplication in the amplitude (if any), is shown as follows:

$$y(t) = 2h(t + 2) + h(t - 2)$$

Note that: For the plots of parts (b) and (c), we do the same process.

**Problem 2.40**

(a) $m(t) = x(t) \ast y(t) = y(t) \ast x(t) = \int_{-\infty}^{\infty} y(\tau) x(t - \tau) \, d\tau$

Now we consider the value of $m(t)$ over different range of time $t$ by moving $x(t - \tau)$.

**Case 1:** When $t + 1 < 0$, this implies that

$\quad t < -1$

Therefore, $m(t) = 0$

**Case 2:** When $t + 1 < 2$, this implies that

$\quad -1 \leq t < 1$

Therefore,

$$m(t) = \int_{0}^{t+1} d\tau = t + 1$$
Case 3: When $t + 1 < 4$, this implies that

$$1 \leq t < 3$$

Therefore,

$$m(t) = \int_{t-1}^{2} d\tau + \int_{2}^{t+1} 2d\tau = t + 1$$

Case 4: When $t - 1 < 4$, this implies that

$$3 \leq t < 5$$

Therefore,

$$m(t) = \int_{t-1}^{4} 2d\tau = 10 - 2t$$

Case 5: When $t - 1 > 4$, this implies that

$$t \geq 5$$

Therefore, $m(t) = 0$

Finally, we can say that

$$m(t) = \begin{cases} 
0 & t < -1 \\
t + 1 & -1 \leq t < 1 \\
t + 1 & 1 \leq t < 3 \\
10 - 2t & 3 \leq t < 5 \\
0 & t \geq 5 
\end{cases}$$
(g) \[ m(t) = y(t) * g(t) = g(t) * y(t) = \int_{-\infty}^{\infty} g(\tau) y(t - \tau) d\tau \]

Now we consider the value of \( m(t) \) over different range of time \( t \) by moving \( y(t - \tau) \).

**Case 1:** When \( t < -1 \), \( m(t) = 0 \)

![Case 1 Diagram]

**Case 2:** When \( t < 1 \), this implies that 
\[-1 \leq t < 1\]

Therefore, 
\[ m(t) = \int_{-1}^{t} \tau d\tau = 0.5[t^2 - 1] \]

![Case 2 Diagram]

**Case 3:** When \( t-2 < 1 \), this implies that 
\[ 1 \leq t < 3 \]

Therefore, 
\[ m(t) = \int_{-1}^{t-2} 2\tau d\tau + \int_{t-2}^{1} \tau d\tau = 0.5(t-2)^2 - 0.5 \]

![Case 3 Diagram]
Case 4: When \( t - 4 < 1 \), this implies that
\[
3 \leq t < 5
\]
Therefore,
\[
m(t) = \int_{t-4}^{1} 2\tau d\tau = 1 - (t - 4)^2
\]

Case 5: When \( t - 4 \geq 1 \), this implies that
\[
t \geq 5
\]
Therefore, \( m(t) = 0 \)

Finally, we can say that
\[
m(t) = \begin{cases} 
0 & t < -1 \\
0.5[t^2 - 1] & -1 \leq t < 1 \\
0.5(t - 2)^2 - 0.5 & 1 \leq t < 3 \\
1 - (t - 4)^2 & 3 \leq t < 5 \\
0 & t \geq 5
\end{cases}
\]
(h) \[ m(t) = y(t) \ast c(t) = \int_{-\infty}^{\infty} y(\tau) c(t-\tau) \, d\tau \]

Now we consider the value of \( m(t) \) over different range of time \( t \) by moving \( c(t-\tau) \).

Case 1: When \( t + 2 < 0 \), this implies that

\[ t < -2 \]

Therefore, \( m(t) = 0 \)

Case 2: When \( t + 2 < 2 \), this implies that

\[ -2 \leq t < 0 \]

Therefore, \( m(t) = 1 \) (from \( \delta(\tau) \) at \( \tau = t + 2 \))

Case 3: When \( t < 1 \), this implies that

\[ 0 \leq t < 1 \]

Therefore,

\[ m(t) = \int_{0}^{t} d\tau + 2 = t + 2 \]
Case 4: When $t < 2$, this implies that

$$1 \leq t < 2$$

Therefore,

$$m(t) = \int_{t-1}^{t} d\tau + 2 = 3$$

Case 5: When $t-1 < 2$, this implies that

$$2 \leq t < 3$$

Therefore,

$$m(t) = -1 + \int_{t-1}^{2} d\tau + 2 \int_{2}^{t} d\tau = t - 2$$

Case 6: When $t < 4$, this implies that

$$3 \leq t < 4$$

Therefore,

$$m(t) = -1 + 2 \int_{t-1}^{t} d\tau = 1$$

Case 7: When $t-1 < 4$, this implies that

$$4 \leq t < 5$$

Therefore,

$$m(t) = -2 + 2 \int_{t-1}^{4} d\tau = 8 - 2t$$

Case 8: When $t-2 < 4$, this implies that

$$5 \leq t < 6$$

Therefore,

$$m(t) = -2$$
Case 9: When $t-2 \geq 4$, this implies that

$$t \geq 6$$

Therefore, $m(t) = 0$

Finally, we can say that

$$m(t) = \begin{cases} 
0 & t < -2 \\
1 & -2 \leq t < 0 \\
t + 2 & 0 \leq t < 1 \\
3 & 1 \leq t < 2 \\
(t - 2) & 2 \leq t < 3 \\
1 & 3 \leq t < 4 \\
8 - 2t & 4 \leq t < 5 \\
-2 & 5 \leq t < 6 \\
0 & t \geq 6 
\end{cases}$$

**Problem 2.41**

[Refer to textbook, page 67, for the impulse response $h(t)$ of an RC circuit.]

(a) Use convolution to calculate the received signal due to transmission of a single “1” at time $t = 0$. Note that the received waveform extends beyond time $T$ and into the interval allocated for the next bit, $T < t < 2T$. This contamination is called intersymbol interference (ISI), since the received waveform at any time is interfered with by previous symbols.

The impulse response of an $RC$ circuit is $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$. The output of the system is the convolution of the input, $x(t)$, with the impulse response, $h(t)$. 

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Thus,
\[ y_p(t) = h(t) * p(t) = p(t) * h(t) = \int_{-\infty}^{\infty} p(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} p(\tau) \frac{1}{RC} e^{-(t-\tau)/(RC)} d\tau \]

Now we consider the value of \( y_p(t) \) over different range of time \( t \).

**Case 1:** When \( t < 0 \), \( y_p(t) = 0 \)

**Case 2:** When \( t < T \), this implies that \( 0 \leq t < T \)

Therefore,
\[ y_p(t) = \int_0^t \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau \]
\[ y_p(t) = 1 - e^{-\frac{t}{RC}} \]

**Case 3:** When \( t \geq T \), we have
\[ y_p(t) = \int_0^T \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau \]
\[ y_p(t) = e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}} \]

Finally, we can say that
\[ y_p(t) = \begin{cases} 
0 & t < 0 \\
1 - e^{-\frac{t}{RC}} & 0 \leq t < T \\
e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}} & t \geq T 
\end{cases} \]
(b) Use convolution to calculate the received signal due to transmission of the sequences “1110” and “1000”. Compare the received waveforms to the output of an ideal channel \( h(t) = \delta(t) \) to evaluate the effect of ISI for the following choices of \( RC \):

(i) \( RC = 1/T \)
(ii) \( RC = 5/T \)
(iii) \( RC = 1/(5T) \)

Assuming \( T = 1 \)

sequences “1110”

\[
(1) \quad x(t) = p(t) + p(t - 1) + p(t - 2) - p(t - 3) \\
y(t) = y_p(t) + y_p(t - 1) + y_p(t - 2) - y_p(t - 3)
\]

sequences “1000”

\[
(2) \quad x(t) = p(t) - p(t - 1) - p(t - 2) - p(t - 3) \\
y(t) = y_p(t) - y_p(t - 1) - y_p(t - 2) - y_p(t - 3)
\]
2.42. Use the definition of the convolution sum to prove the following properties

(a) Distributive: $x[n] * (h[n] + g[n]) = x[n] * h[n] + x[n] * g[n]$

$$
\text{LHS} = x[n] * (h[n] + g[n])
= \sum_{k=-\infty}^{\infty} x[k] (h[n-k] + g[n-k]): \text{The definition of convolution.}
= \sum_{k=-\infty}^{\infty} (x[k]h[n-k] + x[k]g[n-k]): \text{the dist. property of mult.}
= \sum_{k=-\infty}^{\infty} x[k]h[n-k] + \sum_{k=-\infty}^{\infty} x[k]g[n-k]
= x[n] * h[n] + x[n] * g[n]
= \text{RHS}
$$

(b) Associative: $x[n] * (h[n] * g[n]) = (x[n] * h[n]) * g[n]$

$$
\text{LHS} = x[n] * (h[n] * g[n])
$$
\[ x[n] * \left( \sum_{k=\infty}^{\infty} h[k] g[n - k] \right) \]

\[ = \sum_{l=\infty}^{\infty} x[l] \left( \sum_{k=\infty}^{\infty} h[k] g[n - k - l] \right) \]

\[ = \sum_{l=\infty}^{\infty} \sum_{k=\infty}^{\infty} (x[l]h[k]g[n - k - l]) \]

Use \( v = k + l \) and exchange the order of summation

\[ = \sum_{v=\infty}^{\infty} \left( \sum_{l=\infty}^{\infty} x[l]h[v - l] \right) g[n - v] \]

\[ = \sum_{v=\infty}^{\infty} (x[v] * h[v]) g[n - v] \]

\[ = (x[n] * h[n]) * g[n] \]

\[ = \text{RHS} \]

(c) Commutative: \( x[n] * h[n] = h[n] * x[n] \)

\[ \text{LHS} \quad = \quad x[n] * h[n] \]

\[ = \sum_{k=\infty}^{\infty} x[k]h[n - k] \]

Use \( k = n - l \)

\[ = \sum_{l=\infty}^{\infty} x[n - l]h[l] \]

\[ = \sum_{l=\infty}^{\infty} h[l]x[n - l] \]

\[ = \text{RHS} \]

2.46. Find the expression for the impulse response relating the input \( x[n] \) or \( x(t) \) to the output \( y[n] \) or \( y(t) \) in terms of the impulse response of each subsystem for the LTI systems depicted in

(a) Fig. P2.46 (a)

\[ y(t) = x(t) * \{ h_1(t) - h_4(t) * [h_2(t) + h_3(t)] \} * h_5(t) \]

(b) Fig. P2.46 (b)

\[ y[n] = x[n] * \{-h_1[n] * h_2[n] * h_4[n] + h_1[n] * h_3[n] * h_5[n]\} * h_6[n] \]

(c) Fig. P2.46 (c)

\[ y(t) = x(t) * \{-h_1(t) + h_2(t)\} * h_3(t) * h_4(t) + h_2(t) \]