1. A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.
   (a) Find the sample space.
   \[ S = \{ 6,6 \ 6,5 \ 6,4 \ 6,3 \ 6,2 \ 6,1 \ 5,6 \ 5,5 \ 5,4 \ 5,3 \ 5,2 \ 5,1 \ 4,6 \ 4,5 \ 4,4 \ 4,3 \ 4,2 \ 4,1 \ 3,6 \ 3,5 \ 3,4 \ 3,3 \ 3,2 \ 3,1 \ 2,6 \ 2,5 \ 2,4 \ 2,3 \ 2,2 \ 2,1 \ 1,6 \ 1,5 \ 1,4 \ 1,3 \ 1,2 \ 1,1 \} \]

   (b) Find the set \( A \) corresponding to the event “number of dots in the first toss is not less than number of dots in second toss.”
   \[ A = \{ 6,6 \ 6,5 \ 6,4 \ 6,3 \ 6,2 \ 6,1 \ 5,5 \ 5,4 \ 5,3 \ 5,2 \ 5,1 \ 4,4 \ 4,3 \ 4,2 \ 4,1 \ 3,3 \ 3,2 \ 3,1 \} \]

   (c) Find the set \( B \) corresponding to the event “number of dots in first toss is 6.”
   \[ B = \{ 6,6 \ 6,5 \ 6,4 \ 6,3 \ 6,2 \ 6,1 \} \]

   (d) Does \( A \) imply \( B \) or does \( B \) imply \( A \) 

   \[ \text{B implies A} \]

   (e) Find \( A \cap B \) and describe this event in words
   \[ A \cap B = \{ 5,5 \ 5,4 \ 5,3 \ 5,2 \ 5,1 \ 4,4 \ 4,3 \ 4,2 \ 4,1 \ 3,3 \ 3,2 \ 3,1 \ 2,2 \ 2,1 \ 1,1 \} \]

   This is the event that “number of dots in the first toss is not equal to 6 and not less than number of dots in second toss.”

   (f) Let \( C \) correspond to the event “number of dots in dice differs by 2.” Find \( A \cap C \)
   \[ A \cap C = \{ 6,4 \ 5,3 \ 4,2 \ 3,1 \} \]
2. A number $U$ is selected at random between 0 and 1. Let the events $A$ and $B$ be: 
$A = \text{“}U\text{ differs from 1/2 by more than 1/4} \text{”}$ and $B = \text{“}1-U\text{ is less than 1/2} \text{”}$. Find the events $A \cap B$, $A \cap \bar{B}$, $A \cup B$

**Solution**

Let $U \in [0,1]$.

Since $A = \text{“}U\text{ differs from 1/2 by more than 1/4} \text{”}$,

$$A = \left\{ \left| U - \frac{1}{2} \right| > \frac{1}{4} \right\}$$

$$= \left\{ \left( U - \frac{1}{2} \right) > \frac{1}{4} \right\} \cup \left\{ \left( U - \frac{1}{2} \right) < -\frac{1}{4} \right\}$$

$$= \left\{ U > \frac{3}{4} \right\} \cup \left\{ U < \frac{1}{4} \right\}$$

Since $B = \text{“}1-U\text{ is less than 1/2} \text{”}$,

$$B = \left\{ 1-U < \frac{1}{2} \right\}$$

$$= \left\{ U > \frac{1}{2} \right\}$$

Thus,

$$A \cap B = \left( \frac{3}{4}, 1 \right]$$

$$\bar{A} \cap B = \left[ \frac{1}{4}, \frac{3}{4} \right) \cap \left( \frac{1}{2}, 1 \right] = \left( \frac{3}{4}, \frac{3}{2} \right]$$

$$A \cup B = \left\{ U > \frac{1}{2} \right\} \cup \left\{ U < \frac{1}{4} \right\} \text{ or } \left[ 0, \frac{1}{4} \right) \cup \left( \frac{1}{2}, 1 \right].$$
3. Let the events $A$ and $B$ have $P(A) = x$, $P(B) = y$, and $P(A \cup B) = z$. Use Venn diagrams to find $P(A \cap B)$, $P(\overline{A} \cap \overline{B})$, $P(\overline{A} \cup \overline{B})$, $P(A \cup B)$, $P(\overline{A} \cap \overline{B})$ in terms of $x$, $y$, and $z$.

**Solution**

\[ P(A \cap B) = P(A) + P(B) - P(A \cup B) \]
\[ = x + y - z \]

\[ P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) \]
\[ = 1 - z \]

\[ P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B) \]
\[ = 1 - x - y + z \]
\[ P(\overline{A} \cup B) = P(\overline{A} \cap B) + P[B] \]
\[ = 1 - z + y \]

\[ P(A \cap \overline{B}) = P[A] - P(A \cap B) \]
\[ = z - y \]
4. Problem 2.7
For simplicity we shall ignore outcomes falling exactly on the lines, hence we use > or < only rather than ≥ or ≤.

Matlab Problem:
While vacationing in Hawaii, you met this surfer dude who was also a self-claimed clairvoyant. To prove his point, he asked you to pick a secret number between 0 and 1. He then told you to pick a random number also between 0 and 1, and add this random number to the secret number. Next, he asked you to touch your left ear if this sum is greater or equal to 1, and touch your right ear otherwise. He asked you to repeat the same process, keeping the secret number the same, but each time choose a different random number to add to it. This surfer dude didn’t know your chosen random numbers, but after observing you touch your left and right ears for a while, to your surprise, he guessed your number to an amazing accuracy.

How did he do it? Write a MatLab program to confirm your idea.
**Solution**
Let $s$ be the secret number and $r$ is the random number. By performing $N$ trials, we can find the number of touches of left and right ear, $N(\text{left})$ and $N(\text{right})$, respectively.

\[
N(\text{left}) = N(s + r \geq 1) = N(r \geq 1 - s)
\]
\[
N(\text{right}) = N(s + r < 1) = N(r < 1 - s)
\]

\[
P\{r \geq 1 - s\} = N(\text{left})/N
\]
\[
P\{r < 1 - s\} = N(\text{right})/N
\]

Notice that $P\{r < 1 - s\} = N(\text{right})/N$ is nothing more than the ratio of the length of interval $[0, 1-s)$ divided by the interval $[0, 1]$. Therefore,

\[
P\{r < 1 - s\} = \frac{\text{length of interval } [0, 1-s)}{\text{length of } [0, 1]} = \frac{1-s}{1} = 1 - s
\]

Therefore, we guess the secret number, $s$ as $\text{guess} = 1 - N(\text{right})/N$

**Program**

```matlab
s = 0.79
N = 1000
N_right = 0;
for i=1:N
    r = rand();
    if(s+r<1) % touch right ear
        N_right = N_right+1;
    end;
end;
P_right = N_right/N;
guess = 1-P_right
difference = s-guess
```

**Result**
Suppose we pick $s = 0.79$, the program above guess $= 0.7870$ with difference $= 0.0030$. By increasing $N$, you will have a more accurate guess.