1. Problem 3-9.

(a) Valid – non-decreasing and continuous from the right.

\[ P\{0 < X < 1\} = F_X(1-) - F_X(0) = 0.2(1 + 1) - 0.2(0 + 1) = 0.2 \]
\[ P\{0 < X \leq 1\} = F_X(1) - F_X(0) = 0.5 - 0.2(0 + 1) = 0.3 \]
\[ P\{1 < X < 2\} = F_X(2-) - F_X(1) = 0.5 - 0.5 = 0 \]
\[ P\{1 \leq X < 2\} = F_X(2-) - F_X(1-) = 0.5 - 0.2(1 + 1) = 0.1 \]
\[ P\{1 \leq X \leq 2\} = F_X(2) - F_X(1-) = 0.2(2 + 1) - 0.2(1 + 1) = 0.2 \]
\[ P\{X = 2\} = F_X(2) - F_X(2-) = 0.2(2 + 1) - 0.5 = 0.1 \]

(b) Not valid, since it is not continuous from the right.
(c) Not valid, since it rises to 2 at \( u = 2 \) and decreases back to 1.
(d) Valid and is continuous with no jumps.

\[ P\{0 < X < 1\} = P\{0 < X \leq 1\} = F_X(1) - F_X(0) = \frac{1}{2} \]
\[ P\{1 < X < 2\} = P\{1 \leq X < 2\} = P\{1 \leq X \leq 2\} = F_X(2) - F_X(1) = 1 - \frac{1}{2} = 0.5 \]
\[ P\{X = 2\} = 0 \quad \text{No jumps} \]
2. Problem 3-10.

(a) Not valid – negative in the range \(1 < u < 2\)

(b) Valid

\[
1 = \int_0^1 C \sin(\pi u) du = \frac{C}{\pi} \cos(\pi u) \bigg|_0^1 = \frac{2C}{\pi} \Rightarrow C = \frac{\pi}{2}
\]

\[
P\{0 < X < 0.5\} = \int_0^{0.5} \frac{C}{2} \sin(\pi u) du = \frac{1}{2} \cos \pi u \bigg|_0^{0.5} = 0.5
\]

\[
P\{0 < X \leq 0.5\} = P\{0 < X < 0.5\} = 0.5, \quad \text{(since it is continuous)}
\]

\[
P\{X = 0.6\} = 0, \quad \text{(since it is continuous)}
\]

(c) Valid

\[
1 = \int_0^1 C \exp(-2u) du = \frac{C}{2} \exp(-2u) \bigg|_0^1 = C \frac{1-e^{-2}}{2}
\]

\[
\Rightarrow C = \frac{2}{1-e^{-2}} = 2.313
\]

\[
P\{0 < X < 0.5\} = P\{0 < X \leq 0.5\} = \int_0^{0.5} \frac{2}{1-e^{-2}} \exp(-2u) du
\]

\[
= \frac{1-e^{-1}}{1-e^{-2}} = \frac{1}{1+e^{-1}} = 0.731
\]

\[
P\{X = 0.6\} = 0, \quad \text{(since it is continuous)}
\]

(d) Valid, as it is non-negative

\[
1= \int_0^1 C (1-u) du = C \{0.5[1-1]^2 + 0.5[1-(-1)^2]\} = 2C. \quad \text{Hence C=0.5}
\]

\[
P\{0 < X \leq 0.5\} = P\{0 < X < 0.5\} = 0.5 \int_0^{0.5} (1-u) du = 3/16, \quad \text{(since it is continuous)}
\]

\[
P\{X = 0.6\} = 0, \quad \text{(since it is continuous)}
\]

(e) Not valid, as it is negative in the range \(1 < u < 2\)

3. Modeling real-world problem with probability theory.

Give examples of real-world phenomenon that can be reasonably modeled by the following pmfs. In each case, state the experiment, the sample space, the range of the random variable, the pmf of the random variable:

(a) Bernoulli

(b) Binomial

Make up your own examples. Don't copy samples from the text.
4. **ARQ** (Automatic ReQuest) Protocol. A source wishes to transmit data packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is received error-free, the receiver sends an acknowledgment (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgment (NAK) message is sent back to the source. Each time the source receives a NAK, the packet is retransmitted. We assume that each packet transmission is independently corrupted by errors with probability $q$ and the ACK/NAK messages are also correct.

(a) Find the pmf of $X$, the number of times that a packet is transmitted by the source.

(b) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgment message (ACK or NAK) before retransmitting. Let $T$ equal the time required until the packet is successfully received. What is the relationship between $T$ and $X$? What is the pmf of $T$?

**Solution**

(a) The number of times that a packet is transmitted by the source $X$ is following the geometric distribution, then

$$P(X = k) = q^{k-1}(1-q)$$

for $k \geq 1$

(b) $T = 2X - 1$ [ms]

$$P(T = m) = \begin{cases} q^{\frac{m-1}{2}}(1-q) & \text{for } m \text{ odd and positive} \\ 0, & \text{otherwise} \end{cases}$$

5. **Marathon.**

Every day you consider going jogging. Before each mile, including the first, you will quit with probability $q$, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let $M$ equal the number of miles that you run on an arbitrary day. What is $P(M > 0)$? Find the pmf $P_M(m)$.

(b) Let $r$ be the probability that you run a marathon on an arbitrary day. Find $r$.

(c) Let $J$ be the number of days in one year (not a leap year) in which you run a marathon. Find the pmf $P_J(j)$. This answer may be expressed in terms or $r$ found in part (b).

(d) Define $K = M - 26$. Let $A$ be the event that you have run a marathon. Find $P_{K|A}(k)$.

**Solution**

(a) $P(M > 0) = 1 - q$
\[ P_M(m) = (1 - q)^m q \quad \text{for } m = 0, 1, 2, \ldots \]

(b) You run a marathon whenever you run at least 26 miles, therefore

\[
    r = \sum_{m=26}^{\infty} (1 - q)^m q = (1 - q)^{26} q \sum_{m=0}^{\infty} (1 - q)^m = (1 - q)^{26} q \frac{1}{1 - (1 - q)} = (1 - q)^{26}
\]

(c) \( J \) is following a binomial distribution with probability \( r \), therefore

\[
    P_J(j) = \binom{365}{j} r^j (1 - r)^{365-j}
\]

(d) \( P(A) = r = (1 - q)^{26} \)

\[
    P_{K|A}(k) = \frac{P_{K,A}(k)}{P(A)} = \frac{P_M(k + 26)}{P(A)} = \frac{(1 - q)^{k+26} q}{(1 - q)^{26}} = (1 - q)^k q
\]
6. The Sixers and the Celtics (FYI: One of the best rivals in the history of NBA).
The Sixers and the Celtics play a best out of five playoff series. The series ends as soon
as one of the teams has won three games. Assume that either team is equally likely to win
any game independently of any other game played. Find
(a) The pmf $P_N(n)$ for the total number $N$ of games played in the series;
(b) The pmf $P_W(w)$ for the number of $W$ of Celtic wins in the series;
(c) The pmf $P_L(l)$ for the number $L$ of Celtic losses in the series.

Solution

(a) $n = 3, 4, 5$

A team might win a game with probability $p = \frac{1}{2}$

$$P_N(n = 3) = p^3 + (1 - p)^3 = \frac{1}{4}$$

$$P_N(n = 4) = \binom{3}{2} p^3 (1 - p) + \binom{3}{2} p^2 (1 - p)^2 = \frac{3}{8}$$

$$P_N(n = 5) = \binom{4}{2} p^3 (1 - p)^2 + \binom{4}{2} p^2 (1 - p)^3 = \frac{3}{8}$$

(b) $w = 0, 1, 2, 3$

$$P_w(w = 0) = (1 - p)^3 = \frac{1}{8}$$

$$P_w(w = 1) = \binom{3}{1} p(1 - p)^3 = \frac{3}{16}$$

$$P_w(w = 2) = \binom{4}{2} p^2 (1 - p)^3 = \frac{3}{16}$$

$$P_w(w = 3) = p^3 + \binom{3}{2} p^3 (1 - p) + \binom{4}{2} p^2 (1 - p)^2 = \frac{1}{2}$$

(c) $l = 0, 1, 2, 3$

$$P_L(l = 3) = P_w(w = 3) = \frac{1}{2}$$

$$P_L(l = 2) = \binom{4}{2} p^3 (1 - p)^2 = \frac{3}{16}$$

$$P_L(l = 1) = \binom{3}{1} p^3 (1 - p) = \frac{3}{16}$$

$$P_L(l = 0) = p^3 = \frac{1}{8}$$
7. Lottery.
Suppose each day (starting on day 1) you buy one lottery ticket with probability 1/2; Otherwise, you buy no tickets. A ticket is a winner with probability $p$ independent of the outcome of all other tickets. Let $N_i$ be the event that on day $i$ you do not buy a ticket. Let $W_i$ be the event that on day $i$, you buy a winning ticket. Let $L_i$ be the event that on day $i$ you buy a losing ticket.
(a) What are $P(W_{33})$, $P(L_{87})$, and $P(N_{99})$?
(b) Let $K$ be the number of the day on which you buy your first lottery ticket. Find the pmf $P_K(k)$.
(c) Find the pmf of $R$, the number of losing lottery tickets you have purchased in $m$ days.
(d) Let $D$ be the number of the day on which you buy your $j^{th}$ losing ticket. What is $P_D(d)$? Hint: if you buy your $j^{th}$ losing ticket on day $d$, how many losers did you have after $d - 1$ days?

Solution
(a) $P(W_{33}) = p/2$
$P(L_{87}) = (1-p)/2$
$P(N_{99}) = 1/2$

(b) $P_K(k) = \left( \frac{1}{2} \right)^{k-1} \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^k$

(c) $P_R(r) = \binom{m}{r} \left( \frac{1-p}{2} \right)^r \left( \frac{1+p}{2} \right)^{m-r}$

(d) $P_D(d) = \binom{d-1}{j-1} \left( \frac{1-p}{2} \right)^{j-1} \left( \frac{1+p}{2} \right)^{d-j} = \binom{d-1}{j-1} \left( \frac{1-p}{2} \right)^j \left( \frac{1+p}{2} \right)^{d-j}$

8. Coin Toss Take Two. (Bonus problem)
We toss $n$ coins, and each one shows heads with probability $p$, independently of each of the others. Each coin which shows heads is tossed again. What is the pmf of the number of heads resulting from the second round of tosses?

Solution
The probability that showing a head in the second toss is $p^2$
Let $H_2$ be the number of heads from the second round of tosses, then

$$P(H_2 = k) = \binom{n}{k} p^{2k} \left( 1 - p^2 \right)^{n-k} \text{ where } k = 0..n$$