1. Radioactive fruitcake. A Christmas fruitcake has Poisson-distributed independent numbers of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits with respective averages 48, 24, and 12 bits per cake. Suppose you politely accept 1/12 of a slice of the cake.

(a) What is the probability that you get lucky and get no green bits in your slice?

(b) What is the probability that you get really lucky and get no green bits and two or fewer red bits in your slice?

(c) What is the probability that you get extremely lucky and get no green or red bits and more than five raisins in your slice?

Solution:

Let $S$, $R$ and $G$ be the random variable showing the number of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits respectively.

We know $t = 1/12$

(a) $P(G = 0) = \frac{\lambda_G t^0}{0!} e^{-\lambda_G t} = e^{-t}$

(b) $P(G = 0 \land R \leq 2) = P(G = 0) P(R \leq 2) = e^{-t} P(R \leq 2)$

$P(R \leq 2) = P(R = 0) + P(R = 1) + P(R = 2)$

$= \frac{\lambda_R t^0}{0!} e^{-\lambda_R t} + \frac{\lambda_R t^1}{1!} e^{-\lambda_R t} + \frac{\lambda_R t^2}{2!} e^{-\lambda_R t}$

$= e^{-2}(1 + 2 + 2) = 5e^{-2}$

Therefore,

$P(G = 0 \land R \leq 2) = 5e^{-3}$

(c) $P(G = 0 \land R = 0 \land S > 5) = P(G = 0) P(R = 0) P(S > 5)$

$P(R = 0) = \frac{\lambda_R t^0}{0!} e^{-\lambda_R t} = e^{-2}$
\[ P(S > 5) = 1 - P(S \leq 5) \]
\[ = 1 - e^{-\lambda t} \left( \frac{\lambda^0 t^0}{0!} + \frac{\lambda^1 t^1}{1!} + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \frac{\lambda^4 t^4}{4!} + \frac{\lambda^5 t^5}{5!} \right) \]
\[ = 1 - e^{-4} \left( 1 + 4 + 8 + \frac{32}{3} + \frac{32}{3} + \frac{128}{15} \right) \]
\[ = 1 - \frac{643}{15} e^{-4} \]

\[ P(G = 0 \cap R = 0 \cap S > 5) = e^{-3} \left( 1 - \frac{643}{15} e^{-4} \right) \]

2. Impatient taxi driver. Passengers arrive at a taxi stand at an airport at the rate of one passenger per minute follow a Poisson distribution. The taxi driver will leave if either 4 passengers arrive to fill his cab or 5 minutes have passed, whichever occurs first.
(a) Find the probability that the taxi driver will take off without any passenger.
(b) Find the probability that the taxi driver will take off with exactly 4 passengers.

**Solution**
Let random variable \( X \) denote the number of arrived passengers, then
(a)
\[ P_a = P(X = 0) = \frac{\lambda t}{0!} e^{-\lambda t} = \frac{5^0}{0!} e^{-5} = e^{-5} \]
(b)
\[ P_b = P(X \geq 4) = 1 - \sum_{i=0}^{3} P(X = i) = 1 - e^{-5} \left( 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} \right) = 1 - \frac{118}{3} e^{-5} \]

3. Web access. The web accesses to the EECS website can be divided into two types: the internal web access from a computer within OSU network and the external web accesses from a computer outside OSU network. Assuming that the internal and external web accesses follow independent Poisson distributions with the rates of \( \lambda_1 \) and \( \lambda_2 \) hits per minute, respectively. Let \( X \) be the number of hits from either internal or external web accesses.
(a) Show that \( X \) is also a Poisson random variable, and find the new \( \lambda \) in terms of \( \lambda_1 \) and \( \lambda_2 \)?
(b) If \( \lambda_1 = 10 \) and \( \lambda_2 = 20 \), what is the probability that the EECS website has no hit during the next minute?

**Solution**
Let \( I \) and \( E \) show the number of hits for internal and external access, respectively.
(a)
\[ P(X = k) = \sum_{i=0}^{k} P(I = i)P(E = k - i) = \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} e^{-\lambda_1} \frac{\lambda_2^{k-i}}{(k-i)!} e^{-\lambda_2} \]

\[ = e^{-(\lambda_1 + \lambda_2)} \sum_{i=0}^{k} \frac{1}{i!} \frac{\lambda_1^i \lambda_2^{k-i}}{(k-i)!} \]

\[ = e^{-(\lambda_1 + \lambda_2)} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!} \lambda_1^i \lambda_2^{k-i} \]

\[ = e^{-(\lambda_1 + \lambda_2)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} \]

\[ = e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!} \]

\[ = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{where} \quad \lambda = \lambda_1 + \lambda_2 \]

Therefore \( X \) is also a Poisson random variable with rate \( \lambda = \lambda_1 + \lambda_2 \)

(b) \( P(X = 0) = e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^0}{0!} = e^{-30} \)

4. A random variable \( X \) has the pdf:

\[ f_X(x) = \begin{cases} \frac{c}{x(1-x^2)} & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

(a) Find \( c \)

(b) Find \( P(0 < X < 0.5), P(X = 1), P(0.25 < X < 0.5) \)

**Solution:**

(a) We know

\[ \int_{-\infty}^{\infty} f_X(x)dx = 1 = \frac{c}{x(1-x^2)} \bigg|_{0}^{1} = \frac{c}{4} \]

\[ \Rightarrow c = 4 \]

Therefore,

\[ f_X(x) = \begin{cases} 4x(1-x^2), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

(b)

\[ P(0 < x < 0.5) = \int_{0}^{0.5} f_X(x)dx = \int_{0}^{0.5} 4x(1-x^2)dx = 2x^2 - x^4 \bigg|_{0}^{0.5} = \frac{7}{16} = 0.4375 \]

\[ P(X = 1) = F_X(1) - F_X(1^-) = 0 \text{ since } f_X(x) \text{ is continuous at } X = 1 \]

\[ P(0.25 < x < 0.5) = \int_{0.25}^{0.5} f_X(x)dx = \int_{0.25}^{0.5} 4x(1-x^2)dx = 2x^2 - x^4 \bigg|_{0.25}^{0.5} = \frac{79}{156} = 0.5164 \]
5. Dividing Infinity. Let $X$ be an exponential random variable with parameter $\lambda$
(a) for $d > 0$ and $k$ a nonnegative integer, find $P(kd < X < (k + 1)d)$.
(b) Segment the positive real line into four equiprobable disjoint intervals.
Solution:
(a) The exponential distribution with
$f_X(t) = \lambda e^{-\lambda t}$ and $F_X(t) = 1 - e^{-\lambda t}$ for $t > 0$
Therefore,
$$P(kd < X < (k + 1)d) = F_X((k + 1)d) - F_X(kd) = e^{-\lambda kd} - e^{-\lambda kad - \lambda kd} = e^{-\lambda kd}(1 - e^{-\lambda kd})$$
(b) Assuming that exist four equiprobable disjoint intervals as (0,a), (a,b), (b,c), and (c, $\infty$) which have probability of 0.25.
Consider the first interval (0, a):
$$0.25 = P(0 < X < a) = F_X(a) - F_X(0) = 1 - e^{-\lambda a}$$
$$\Rightarrow e^{-\lambda a} = 0.75$$
$$\Rightarrow a = \frac{-\ln 0.75}{\lambda}$$
For the interval (a,b):
$$0.25 = P(a < X < b) = F_X(b) - F_X(a) = e^{-\lambda a} - e^{-\lambda b} = 0.75 - e^{-\lambda b}$$
$$\Rightarrow e^{-\lambda b} = 0.5$$
$$\Rightarrow b = \frac{-\ln 0.5}{\lambda}$$
Similar for the interval (b,c):
$$0.25 = P(b < X < c) = F_X(c) - F_X(b) = e^{-\lambda b} - e^{-\lambda c} = 0.5 - e^{-\lambda c}$$
$$\Rightarrow e^{-\lambda c} = 0.25$$
$$\Rightarrow c = \frac{-\ln 0.25}{\lambda}$$
And for the last interval (c, $\infty$):
$$P(c < X < \infty) = F_X(\infty) - F_X(c) = 1 - (1 - e^{-\lambda c}) = e^{-\lambda c} = 0.25$$

6. Simple circuit. A DC current source $X$ is connected to a resistor of resistance of 1 Ohm.
Due to measurement errors and sources of uncertainty such as the temperature, the current $X$ (amps) better modeled as a random variable distributed according to $X \sim N(4, 1)$. What is the probability that the voltage across the resistor is between 9 and 11 Volts? Leave your answer in terms of $\Phi(.)$, the CDF of the standard Gaussian random variable.
Solution
We need to find
$$P(9 < X < 11) = \Phi\left(\frac{11 - m_X}{\sigma_X}\right) - \Phi\left(\frac{9 - m_X}{\sigma_X}\right) = \Phi\left(\frac{11 - 4}{1}\right) - \Phi\left(\frac{9 - 4}{1}\right) = \Phi(7) - \Phi(5)$$
7. (Bonus). A chord of the unit circle is picked at random. What is the probability that an equilateral triangle with the chord as base can fit inside the circle if
(a) The perpendicular distance from the chord to the center of the circle is uniform on $[0,1]$?
(b) The acute angle between the chord and a tangent at one of its endpoints is uniform on $[0, \pi / 2]$?

Solution:
(a)
Let $h$ is the perpendicular distance from the chord to the center of the circle
Then, the largest equilateral triangle with the chord as base can fit inside the circle is with $h = 0.5$
$$P_a = P(0.5 \leq h \leq 1) = 0.5$$

(b) Let $\theta$ is the acute angle between the chord and a tangent at one of its endpoints
Then, the largest equilateral triangle with the chord as base can fit inside the circle is with $\theta = \pi / 3$
$$P_b = P(0 \leq \theta \leq \pi / 3) = \frac{\pi / 3}{\pi / 2} = \frac{2}{3}$$