Functions of a Random Variable

Suppose you have $Y = g(X)$. $X$ is a r.v. $\Rightarrow Y$ is a r.v.

Now, can we find $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ or $f_X(x)$?

Ex: $X$ is a number of wins you have at a certain game.

Maybe, let

$Y = 2X$. 
Example: At $t = 109$, the response time of a LTI with a single pole is described by $y = 2e^{-\frac{10}{T}}$, where $T$ is the time constant of the system, which is uniformly distributed in the interval $[1, 2]$. What are the CDF and PDF of $Y$?

Solution: $F_Y(y) = P(Y \leq y) = P\left(2e^{-\frac{10}{T}} \leq y\right)$

$$= P\left(T \leq \frac{10}{\ln\left(\frac{2}{y}\right)}\right) = \frac{10}{\ln\left(\frac{2}{y}\right)} \cdot \frac{\ln\left(\frac{2}{y}\right)}{10} = F_Y(y)$$
\[ F(x) = \begin{cases} x - a & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]

\[ F_y(y) = F_T\left( \frac{10}{\ln(2/y)} \right) = \begin{cases} \frac{10}{\ln(2/y)} & 1 \leq 10/ \ln(2/y) \leq 15 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_y(y) = \frac{F_y(y)}{dy} = \frac{F_T\left( \frac{10}{\ln(2/y)} \right)}{dy} \]

\[ \frac{\ln}{dy} \]

\[ \int_{-\infty}^{\infty} f_T(t) \left( \frac{10}{\ln(2/y)} \right) \frac{dy}{2} \bigg|_{t = \frac{10}{\ln(2/y)}} \]
\[ = \sum_{y} \frac{10}{y \ln \left( \frac{2}{y} \right)} y^{2} \quad \text{for} \quad y \in \left( 2e^{-10}, e^{-5} \right) \]

\[ f_{y}(y) = \frac{f_{x}(x)}{|g'(x)|} \quad \text{for} \quad y = g(x) \]

\[ x = g^{-1}(y) \]

assuming that \( g(x) \) is either monotonically increases or monotonically decreases.
Case 1) Assume that $g(x)$ is monotonically increasing function.

\[ F_Y(Y \leq y^*) = P(Y \leq y^*) = P(X \leq x^*) \Rightarrow \text{Not true} \]

For $g(x)$ - that is not monotonically increasing. For example,

\[ p(Y \leq y^*) = p(X > x^*) \]

For monotonically decreasing function.
\[ P(Y \leq y^*) = P(X \leq x^*) = F_X(x^*) = F_X(g^{-1}(y^*)) \]

\[ F_Y(y) = F_X(g^{-1}(y)) \]

\[ f_Y(y) = \frac{dF_Y(y)}{dy} = F_X(g^{-1}(y)) \frac{dx}{dy} \bigg|_{x = g^{-1}(y)} \]

\[ = \frac{f_X(x)}{\frac{dy}{dx}} \bigg|_{x = g^{-1}(y)} = \frac{f_X(x)}{dg(x)/dx} \bigg|_{x = g^{-1}(y)} \]
Example: \( y = -5x + 10 \), and \( X \) be a Gaussian r.v. \( X \sim N(0, 1) \). Find \( f_y(y) \)?

\[
f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\]

\[
g'(x) = -5, \quad g^{-1}(y) = \frac{y - 10}{-5} = \frac{y}{5} - 2
\]

\[
f_y(y) = \frac{f_x(x)}{|g'(x)|} \bigg|_{x = g^{-1}(y)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y - 10}{-5}\right)^2} \left| \frac{1}{-5} \right| = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y}{5} - 2\right)^2}