Random Variable

**Definition:** A random variable $X$ is a real-valued function of the elements of a sample space $S$.

\[ X(\zeta) = x \]

$X : \zeta \rightarrow \mathbb{R}$, $X(\cdot)$ is a function map.

**Convention:** We will use capital letter to represent random variables (r.v.) and lower case letters to represent the values that the r.v.'s take on.
Discrete and Continuous Random Variables

**Definition**  A r.v. is called discrete if its range $R_x$ is a discrete set of real numbers. A r.v. $X$ is called continuous if its range $R_x$ is an interval or union of intervals on the real line and if it has probability zero of equaling any single value in $R_x$.

**Example:** We roll a pair of fair dice 1 time. Let $X$ be the sum of the 2 numbers that occur.

$$R_x = \{2, 3, 4, 5, \ldots, 12\}$$

$$P(X = x) =$$

Discrete random variable
Discrete and Continuous Random Variables

**Example:** A bus stops at station every day at some random time between 11 AM. and 11:30 AM.

If $X$ is the actual arrival time of the bus,

$$X = [11, 11.5]$$

This is a continuous random variable.
Cumulative Distribution Function

**Definition:** Given a sample space $S$ and a r.v. $X(\zeta)$, define $F_X(x)$, for any real value $x$ as the cumulative distribution function (CDF) of the r.v. $X$ by

$$F_X(x) = P(\zeta : X(\zeta) \leq x) = P(X(\zeta) \leq x) = P(X \leq x).$$

**Convention:** From now on we shall use the following convention: $\{X(\zeta) \leq x\} \equiv \{X \leq x\}$.

Let's go back to the rolling a pair of dice example

$$F_X(3) = P(X \leq 3) = \frac{2}{11}$$

$$F_X(4) = \frac{3}{11}$$
Cumulative Distribution Function

Properties of the CDF

1. $F_X(x)$ is nondecreasing, i.e., if $a < b$, then $F_X(a) \leq F_X(b)$. This is true because

$$\{x \leq a\} \Rightarrow \{x \leq b\} \Rightarrow \{x \leq a\} \subseteq \{x \leq b\} \Rightarrow P(x \leq a) \leq P(x \leq b) \Rightarrow F_X(a) \leq F_X(b).$$

2. $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.

Since the event $\{X \leq -\infty\}$ is null (an empty set), then $P(X \leq -\infty) = F_X(-\infty) = 0$. On the other hand, the event $\{X \leq \infty\}$ is the certain event $\Rightarrow P(X \leq \infty) = F_X(\infty) = 1$.

3. $F_X$ is right continuous, i.e., $F_X(a + \varepsilon) = F_X(a)$ as $\varepsilon \to 0$, $\varepsilon > 0$.

Remarks: A distribution function is a monotonic nondecreasing function of its argument starting at 0 and ending at 1. It may have jumps, but at the jumps the value of the distribution is equal to the value at the top of the jump.
Cumulative Distribution Function

In general,

\[ P(X \leq a) = F_X(a) \]
\[ P(X > a) = 1 - P(X \leq a) = 1 - F_X(a) \]
\[ P(X < a) = F_X(a^-) \]
\[ P(X \geq a) = 1 - P(X < a) = 1 - F_X(a^-) \]
\[ P(X = a) = F_X(a) - F_X(a^-) \]
\[ P(a < X \leq b) = F_X(b) - F_X(a) \]
\[ P(a < X < b) = F_X(b^-) - F_X(a) \]
\[ P(a \leq X \leq b) = F_X(b) - F_X(a^-) \]
\[ P(a \leq X < b) = F_X(b^-) - F_X(a^-) \]
**Cumulative Distribution Function**

**Example:** The distribution of a r.v. $X$ is described by

$$
F_X(x) = \begin{cases} 
0, & x < 0 \\
\frac{x}{4}, & 0 \leq x < 1 \\
\frac{1}{2}, & 1 \leq x < 2 \\
\frac{x}{12} + \frac{1}{2}, & 2 \leq x < 3 \\
1, & x \geq 3 
\end{cases}
$$

\[
\frac{1}{4} + \frac{1}{2} = \frac{3}{4}
\]
\[
\frac{1}{6} + \frac{1}{2} = \frac{2}{3}
\]
Cumulative Distribution Function

Compute  

\[ a) P(X < 2), \quad b) P(X = 2), \quad c) P(X = \frac{5}{2}), \quad d) P(2 < X \leq 7) \]

**Solution:**

\[ a) P(X < 2) = F_X(2^-) = \frac{1}{2} \]

\[ b) P(X = 2) = F_X(2) - F_X(2^-) = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6} \]

\[ c) P(X = \frac{5}{2}) = 0 \]

\[ d) P(2 < X \leq 7) = F_X(3) - F_X(2) = 1 - \frac{2}{3} = \frac{1}{3} \]