1. (15pts) The number of packets arriving at a router follows a Poisson random variable with \( \lambda = 1 \). Let \( X \) be a random variable denoting the inter-arrival times between two packets. Let \( Y = e^{-X^2} \) be a random variable denoting the work load of the router. Find \( F_Y(y) \) and \( f_Y(y) \).

\[
F_Y(y) = P\{Y \leq y\} = P\{e^{-X^2} \leq y\} = P\{-X^2 \leq \ln y\} = P\{X^2 \geq \ln \frac{1}{y}\} = P\{X \geq \sqrt{\ln \frac{1}{y}}\}
\]

\( X \) is an exponential random variable with \( \lambda = 1 \). The range of \( X \) is 0 to \( \infty \).

\[
F_Y(y) = P\{X \geq \sqrt{\ln \frac{1}{y}}\} = 1 - \left( P\{X \leq \sqrt{\ln \frac{1}{y}}\} \right) = 1 - \left( 1 - e^{-\sqrt{\ln \frac{1}{y}}} \right) = e^{-\sqrt{\ln \frac{1}{y}}}, 0 \leq y \leq 1
\]

The density function is

\[
f_Y(y) = \frac{d}{dy} e^{-\sqrt{\ln \frac{1}{y}}} = e^{-\sqrt{\ln \frac{1}{y}}} \cdot \frac{-1}{2\sqrt{\ln \frac{1}{y}}} \cdot \frac{1}{y^2} = -\frac{e^{\sqrt{\ln \frac{1}{y}}}}{2\sqrt{\ln \frac{1}{y}}} \cdot \frac{1}{y^2}, 0 \leq y \leq 1
\]

2. A fair coin is flipped twice. Let \( X \) and \( Y \) be two Bernoulli random variables associated with the outcomes of first and second flips, respectively. Let \( Z = X - Y \), and \( W = X + Y \).

a) (15 pts) Determine \( p_{W,Z}(w,z) \) the joint pmf of \( W \) and \( Z \) in the form of a table.

The possible values of \( X \) and \( Y \) and the corresponding \( Z \) and \( W \) are as follows:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>W</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>
The joint pmf is

<table>
<thead>
<tr>
<th>W = 0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z = -1</td>
<td>0</td>
<td>¼</td>
</tr>
<tr>
<td>0</td>
<td>¼</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>¼</td>
</tr>
</tbody>
</table>

b) (10 pts) Are $W$ and $Z$ independent? Give a mathematical justification for your answer.

$p_{W,Z}(w=0,z=0)=1/4$

$p_w(w=0) = ¼$ (By adding the column corresponding to $w = 0$)

$p_z(z=0) = ½$ (By adding the row corresponding to $z = 0$)

$p_{w,z}(w=0,z=0) \neq p_w(w=0)p_z(z=0)$

Therefore $W$ and $Z$ are not independent.

c) (10 pts) What is $\text{Var}[W+2Z]$?

\[
\text{Var}[W + 2Z] = \text{Var}[X + Y + 2X - 2Y] = \text{Var}[3X-Y] = 9\text{Var}[X] + \text{Var}[Y] - 6\text{E}[X]\text{E}[Y] + 6\text{E}[X]\text{E}[Y]
\]

The last equality is true since $X$ and $Y$ are independent.

\[
\text{Var}[X] = \text{Var}[Y] = E[X^2] - E[X]^2
\]

\[
= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
\]

$\text{Var}[W + 2Z] = \frac{9}{4} + \frac{1}{4} = 2.5$

3. The joint pdf of two random variables $X$ and $Y$ is

\[f_{X,Y}(x,y) = Ce^{-(x+y)}, -1 \leq x \leq 1 \text{ and } 0 \leq y \leq x^2\]

a) (10 pts) Determine $C$. 

\[
\int_{-1}^{1} \int_{0}^{2} e^{-x^2} \, dy \, dx = 1
\]

\[
\int_{-1}^{1} e^{-x} \int_{0}^{2} e^{-y} \, dy \, dx = \int_{-1}^{1} e^{-x} \left(1 - e^{-x^2}\right) \, dx = C \left( \int_{-1}^{1} e^{-x} \, dx - \int_{-1}^{1} e^{-\left(x^2 + x\right)} \, dx \right)
\]

\[
= C \left( e^{-1} - \int_{-1}^{1} e^{-\left(x^2 + x\right)/2} \, dx \right)
\]

\[
= C \left( e^{-1} - e^{-x^2} \int_{-1/2}^{1/2} e^{-x^2} \, dx \right)
\]

\[
= C \left( e^{-1} - e^{-1} \left( \Phi \left(1/\sqrt{2} + 1/2\right) - \Phi \left(-1/\sqrt{2} + 1/2\right) \right) \right) = 1
\]

\[
C = \frac{1}{e^{-1} - e^{-1} \left( \Phi \left(1/\sqrt{2} + 1/2\right) - \Phi \left(-1/\sqrt{2} + 1/2\right) \right)}
\]

\[\text{b) (20 pts) Find } f_{X|Y}(x|y).\]

\[
f_{Y}(y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) \, dx = Ce^{-y} \left( e^{-\sqrt{y} - \frac{1}{e} - e + e^{-\sqrt{y}}} \right) = Ce^{-y} \left( 2e^{-\sqrt{y}} - \frac{1}{e} - e \right), 0 \leq y \leq 1
\]

\[
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_{Y}(y)} = \frac{e^{-x}}{2e^{-\sqrt{y}} - \frac{1}{e} - e}, -1 \leq x \leq 1
\]

\[\text{c) (20 pts) Find the } P(X < \frac{1}{2}, Y < 1/2)\]

\[
\int_{-1}^{1/2} \int_{0}^{1/2} e^{-y} \, dy \, dx + \int_{-1/2}^{1} \int_{0}^{1/2} e^{-y} \, dy \, dx = \int_{-1/2}^{1/2} \int_{0}^{1/2} e^{-y} \, dy \, dx + \int_{-1/2}^{1/2} \int_{0}^{1/2} e^{-\left(x^2 + x\right)} \, dy \, dx
\]

\[
= C \left( 1 - e^{-1/2} \right) \left( e - e^{-1/4} \right) + C \left( \int_{-1/2}^{1/2} e^{-x} \, dx - \int_{-1/2}^{1/2} e^{-\left(x^2 + x\right)} \, dx \right)
\]

\[
= C \left( 1 - e^{-1/2} \right) \left( e - e^{-1/4} \right) + C \left( e^{1/4} - e^{-1/4} - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} e^{-\left(x^2 + x\right)/4} \, dx \right)
\]

\[
= C \left( 1 - e^{-1/2} \right) \left( e - e^{-1/4} \right) + C \left( e^{1/4} - e^{-1/4} - e^{-\frac{1}{4}} \int_{-1/\sqrt{2} + 1/2}^{1/\sqrt{2} + 1/2} e^{-x^2} \, dx \right)
\]

\[
= C \left( 1 - e^{-1/2} \right) \left( e - e^{-1/4} \right) + C \left( e^{1/4} - e^{-1/4} - e^{-\frac{1}{4}} \left( \Phi \left(1/\sqrt{2} + 1/2\right) - \Phi \left(-1/\sqrt{2} + 1/2\right) \right) \right)
\]
4. Bonus Problem (10pts) Suppose you want to generate a samples $y$ from a continuous random variable $Y$ with a given cumulative distribution $F_Y(y)$ with $F_Y(y)$ being a monotonically increasing function (note that in general, $F_Y(y)$ must be a non-decreasing function, but not necessarily a monotonically increasing function.) Show that you can do that by a) generating a sample $x$ uniformly at random between 0 and 1 and b) letting $y = F_Y^{-1}(x)$.

If we do as above, the cdf of the resulting random variable is

$$P\{Y \leq y\} = P\{F_Y^{-1}(X) \leq y\} = P\{X \leq F_Y(y)\}$$

The inverse is unique since $F_Y(y)$ is a monotonically increasing function. Now, $X$ is a uniform random variable and the above probability is given by

$$P\{Y \leq y\} = P\{X \leq F_Y(y)\} = F_Y(y)$$