Random Markov sequence

Definition: a) Continuous-valued Markov random sequence $X[n]$ defined for $n \geq 0$, satisfies the conditional pdf

$$f_X(x_{n+k} | x_n, x_{n-1}, \ldots, x_0) = f_X(x_{n+k} | x_n)$$

b) Discrete-valued Markov random sequence $X[n]$ for $n \geq 0$, satisfies the conditional pmf

$$P_X(x_{n+k} | x_n, x_{n-1}, \ldots, x_0) = P_X(x_{n+k} | x_n)$$

Note: It is easier to compute the joint probability of a random Markov sequence.

$$f_X(x_0, x_1, \ldots, x_n) = f_X(x_0) f_X(x_1 | x_0) \ldots f_X(x_n | x_{n-1})$$

$$= f_X(x_0) f_X(x_1 | x_0) f_X(x_2 | x_1, x_0) \ldots f_X(x_n | x_{n-1}, x_{n-2}, \ldots, x_0)$$

$$= f_X(x_0) f_X(x_1 | x_0) f_X(x_2 | x_1, x_0) f_X(x_3 | x_2, x_1, x_0) \ldots f_X(x_n | x_{n-1}, \ldots)$$
Unify Markov properties

\[= \prod_{i=1}^{n} \mathcal{N}(x_i | x_{i-1}) \]

Markov

Ex: let \(X \in \mathbb{N}^T\) be a random sequence defined for \(n > 1\)

\[\mathcal{N}(x) = \mathcal{N}(0, \delta^2)\]

and the conditioned pdfs

\[\mathcal{N}(x_n | x_{n-1}) = \mathcal{N}(x_n | \mu, \sigma^2)\]

Determine \(\mathcal{N}(x) = \mathcal{N}(0, \delta^2)\)

\[\mathcal{N}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathcal{N}(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots \, dx_n\]

(1)
you pluggin each \( f_X(x_n|x_{n-1}) \) to find the answer (3). Remember \( E[Y] = E[E[Y|X]] \)?

\[
\begin{align*}
\mu_{X[n]} &= E[X[n]] \\
&= E[E[X[n] | X[n-1]]] \\
&= E(\phi X[n-1]) = \phi \mu_{X[n-1]} = \phi^2 \mu_{X[n-2]} \\
&= + \ldots = \phi^n \mu_{X[0]} = 0
\end{align*}
\]

\[
\begin{align*}
E[X^2[n]] &= E[E[X^2[n] | X[n-1]]] \\
&= E(\phi^2 + \phi^2 X[n-1]) \\
&= \phi^2 + \phi^2 E[X^2[n-1]] \\
&= \phi^2 + \phi^2 \left( \phi^2 + \phi^2 E[X^2[n-2]] \right) \\
&= \phi^2 + \phi^4 \phi^2 + \phi^4 E[X^2[n-2]] \\
&= \phi^2 \left( 1 + \phi^2 + \phi^4 + \ldots + \phi^{2(n-1)} \right) \mu_{X[0]} + \phi^{2n} \mu_{X[0]}
\end{align*}
\]
\[ E \left[ x_{\infty}^2 \right] \xrightarrow{n \to \infty} \frac{2w}{1-e^{-2}} \]

**Markov Chain:**

**Definition:** A discrete-time Markov chain is a random sequence \( X[n] \) whose \( N \)th order conditional pmf satisfies

\[
p_X \left( x_n, x_{n-1}, \ldots, x_{n-N} \right) = p_X \left( x_n \mid x_{n-1} \right), \quad \forall n, \quad N > 1.
\]
Typically, a Markov chain can be represented by a diagram with a transition probability matrix $P$. Let $X[n] \in \{1, 2\}$.

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$X[0] = 1$ \text{ stay}

$X[1] = 1$ \text{ stay}

$X[2] = 2$ \text{ Jump}

$X[3] = 1$ \text{ Jump}
Theorem: If a Markov chain is irreducible and aperiodic, then there exists a unique stationary distribution \( \pi = \left[ \pi_1, \pi_2, \ldots, \pi_n \right] \) such that

\[
\lim_{{n \to \infty}} \pi^T \mathbf{P}^n = \pi^T
\]

for any initial distribution \( \pi \).

Irreducible

Every state can be reached by any other state.

Reducible

State 4 cannot reach state 1.
Aperiodic

\[ p_{12} \neq 1 \]

\[ p_{21} \neq 1 \]

\[ \ldots 1, 2, 1, 2, 1, 2, \ldots \]

\[ \ldots 1, 2, 1, 2, 1, 2, \ldots \]

\[ \begin{align*}
V_1^T &= V_0^T P \\
V_n^T &= V_{n-1}^T P = V_{n-2}^T P^2 = \ldots = V_0^T P^n, \quad n \to \text{large} \Rightarrow \pi^T
\end{align*} \]

If MC is irreducible and aperiodic.

Note: \( \pi^T P = \pi^T \)
Ex: Suppose there's frog on Lily pad #1. It jumps to another Lily pad with probability 0.1, and stays in the same Lily pad with probability 0.9. When it is in Lily pad #2, it will jump back to Lily pad #1 with probability 0.2, and stays with the Lily pad #2 with probability 0.8.

Question: After a year, you want to know which Lily pad it stays on.

Computing stationary distribution:

\[ P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]
\[
\begin{align*}
\mathbf{\Pi} = \mathbf{\Pi}_P &= \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \\
&= \begin{bmatrix} (\pi_1 \cdot 0.9) + \pi_2 \cdot (0.2) \\ \pi_1 (0.1) + \pi_2 (0.8) \end{bmatrix}
\end{align*}
\]

\[
\Rightarrow \quad \pi_1 = \pi_1 (0.9) + \pi_2 (0.2) \\
\pi_1 + \pi_2 = 1
\]

\[
\Rightarrow \quad \pi_1 + 0.5 \pi_1 = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{1.5} = \frac{2}{3}
\]

\[
\pi_2 = \frac{1}{3}
\]
ex: queuing at DMV
Random Process

\[ X(t; w) \]

\[ M_X(t) = \mathbb{E}[X(t)] \]

This is now replaced with:

\[ \varphi_{XX}(t_1, t_2) = \mathbb{E}[X(t_1) X(t_2)] \]

\[ -\infty < t_1 < \infty \]

\[ -\infty < t_2 < \infty \]
ex: Consider a random process

\[ X(t) = A \sin(\omega_0 t + \Theta) \]

where \( A \) and \( \Theta \) are independent RVS, \( \Theta \sim \text{Uni } [-\pi, \pi] \)

compute \( E[X(t)] \) and \( R_{XX}(t_1, t_2) \).

a) \[ E[X(t)] = E[A \sin(\omega_0 t + \Theta)] = E[A] E[\sin(\omega_0 t + \Theta)] \]
\[ = E[A] \int_{-\pi}^{\pi} \sin(\omega_0 t + \Theta) \, d\Theta = E[A] \cdot 0 = 0 \]

b) \[ R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)] = E[A^2 \sin(\omega_0 t_1 + \Theta) \sin(\omega_0 t_2 + \Theta)] \]
\[ = E[A^2] E[\sin(\omega_0 t_1 + \Theta) \sin(\omega_0 t_2 + \Theta)] \]
\[ = E[A^2] E\left[ \frac{1}{2} \cos(\omega_0 (t_1 - t_2)) - \frac{1}{2} \cos(\omega_0 (t_1 - t_2) + 2\Theta) \right] \]
\[
\sin a \sin b = \frac{1}{2} \left( \cos(a-b) + \cos(a+b) \right) \\
= E[A^2] \frac{1}{2} \cos(w_0(t_1-t_2)) = \frac{1}{2} E[A^2] E[\cos(w_0(t_1+t_2+2\theta))] \\
= \frac{1}{2} E[A^2] \cos(w_0(t_1-t_2))
\]

**Definition**

**WSS Process**

\[ M_x(t) = \text{constant} \]
\[ R_{xx}(t_1, t_2) = R_{xx}(t_1-t_2) \]
Poisson counting process

Let $N(t)$ be a process that represents the total number of arrivals up to time $t$. The interarrival time is exponentially distributed.

$N(t)$ is a random process:

$$N(t) = \sum_{n=1}^{\infty} \delta(t - T[n]),$$

where

$T[n]$ : time to $n$th arrival
From past lecture we have

\[ S(t) = \frac{(\lambda t)^{n-1} \lambda e^{-\lambda t}}{(n-1)!} \]

\[ P(N(t) = n) = P(T[n] \leq t, T[n+1] > t) \]

\[ = P(T[n] \leq t, T[n+1] > t - T[n]) \]

Where \( T[n] \) is interarrival time: \( T[n] \sim \lambda e^{-\lambda t} \)

\[ = \int_0^t \left( \int_{t-x}^\infty \frac{(\lambda t)^{n-1} \lambda e^{-\lambda t}}{(n-1)!} \right) \left( \int_{t-x}^\infty \frac{(\lambda x)^{n-1} \lambda e^{-\lambda x}}{(n-1)!} \right) dx \]
\[
\begin{align*}
&= \int_0^t \left( \frac{(\lambda x)^{n-1}}{(n-1)!} \right) \lambda e^{-\lambda x} \int_0^\infty e^{-\beta} \beta^{t-2} d\beta \, dx \\
&= \int_0^t \left( \frac{(\lambda x)^{n-1}}{(n-1)!} \right) \lambda e^{-\lambda x} \left[ -e^{-\beta} \right]_0^\infty \, dx \\
&= (\lambda t)^n e^{-\lambda t} \\
&= \left( \frac{\lambda t}{n!} \right)^n e^{-\lambda t} \quad \text{for } t > 0.
\end{align*}
\]
Definition: An independent increment random process is the one that has, \( X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1}) \) are jointly independent if \( t_1 < t_2 < t_3, \ldots, t_n \geq 1 \).

\( N(t) \) is an independent increment random process because:

\[ N_0(t_1) \equiv N(t_2) - N(t_1) \equiv N(t_3) - N(t_2) \equiv \ldots \]

\# number of arrivals between \( t_0 \) and \( t_1 \)

\# number of arrivals between \( t_3 \) and \( t_2 \)