1 Statistical Specification of a Random Sequence

A random sequence \( X[n] \) is statistically specified by knowing its Nth-order CDFs and PMFs.

**Definition 1.1 Joint Cumulative Distribution Function:**
A Random Sequence \( X[n] \) is Nth-order CDFs for all integers \( N \geq 1 \) and for all times, \( n, n+1...n+N-1 \)

\[
F_X(x_n, x_{n+1}, ..., x_{n+N-1}) \triangleq P(X[n] \leq x_n, X_{n+1} \leq x_{n+1}, ..., X_{n+N-1} \leq x_{n+N-1})
\]

**Definition 1.2 Joint Probability Mass Function:**
The Nth-order probability Mass functions (pmf’s) are given by differentiable \( F_X \) as

\[
F_X(x_n, x_{n+1}, ..., x_{n+N-1}) \triangleq \frac{\partial^n F_x(x_n, x_{n+1}, ..., x_{n+N-1})}{\partial x_n \partial x_{n+1} \cdots \partial x_{n+N-1}}
\]

**Example** \( X[n] = \frac{W}{n} \) where \( W = \begin{cases} 1 & \text{with } p \\ -1 & \text{with } 1-p \end{cases} \)

now, when \( n = 1 \)

\[
X[1] = \begin{cases} 1 & \text{with } p \\ -1 & \text{with } 1-p \end{cases}
\]

when \( n = 2 \)

\[
X[2] = \begin{cases} \frac{1}{2} & \text{with } p \\ -\frac{1}{2} & \text{with } 1-p \end{cases}
\]
Example \( X[n] = \frac{W[n]}{n} \) where \( W = \begin{cases} 1 \text{ with } p \\ -1 \text{ with } 1 - p \end{cases} \)

now, when \( n = 1 \)

\[ X[1] = \begin{cases} 1 \text{ with } p \\ -1 \text{ with } 1 - p \end{cases} \]

when \( n = 2 \)

\[ X[2] = \begin{cases} -\frac{1}{2} \text{ with } p \\ \frac{1}{2} \text{ with } 1 - p \end{cases} \]
For C.R.V

\[ \mu_X[n] \triangleq E[X[n]] = \int_{-\infty}^{\infty} x f_X(x; n) dx \triangleq \int_{-\infty}^{\infty} x^n f_X(x[n]) dx[n] \triangleq \int_{-\infty}^{\infty} x_n f_X(x_n) dx_n \]

For D.R.V

\[ \mu_X[n] = \sum_{k=-\infty}^{\infty} x_k P(X[n] = x_k) = \sum_{k=-\infty}^{\infty} x_k P_X(x_k) \]

**Definition 1.3 Marginal PDF**

\[ f_X(x_i) = \int_{x_1} f_{X_2} dx_2 \cdots \int_{x_{i-1}} f_{X_{i+1}} dx_{i+1} \int f_X(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} dx_n \]

Similarly, for PMF

\[ P_X(x_i) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} P_X(x_1, x_2, \ldots, X_n) \]

**Definition 1.4 Auto Correlation Function**

\[ R_{XX}(k, l) \triangleq E[X[k]X^*[l]] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_k x_l^* f_X(x_k, x_l) dx_k dx_l \]

**Definition 1.5 Auto Covariance Function**

\[ K_{XX}(k, l) \triangleq E[(X[k] - \mu_X[k])(X[l] - \mu_X[l])^*] \]

**Definition 1.6 Properties of** \( K_{XX} \) and \( R_{XX} \)

*Both* \( R_{XX} \) and \( K_{XX} \) *are Hermitian; i.e. they are complex conjugate*

\[ R_{XX}[k, l] = R_{XX}^*[l, k] \]
\[ K_{XX}[k, l] = K_{XX}^*[l, k] \]

**Definition 1.7 Variance Function:**

\[ \sigma_X^2[n] \triangleq K_{XX}[n, n] = K_{XX}[n] \]
2 Correlated Noise

*Correlate Noise* In electric circuit or communication signal, we can see the noise be added in signal feedback, we want to know about $x[n]$, therefore we want to see expect of $x[n]$ and the $\text{var}[x]$. We have a system like picture below:

![Figure 3: Correlate Noise](image)

Note that: $W[n]$ is $\text{Bern}(p)$, and is i.i.d. distribution:

$$W[n] = \begin{cases} 0, & \text{with probability } (1 - p) \\ 1, & \text{with probability } (p) \end{cases}$$

We have:

$$X[0] = W[0]$$

General case:

$$X[n] = \sum_{k=0}^{k=n} \alpha^{n-k} W[k]$$

2.1 Compute the expectation $E[X[n]]$

Therefore, we can compute the means of $X[n]$ as:

$$E(X[n]) = \mu_X[n] = E\left[\sum_{k=0}^{k=n} \alpha^{n-k} W[k]\right] = \sum_{k=0}^{k=n} \alpha^{n-k} p = p \frac{\alpha^{n+1} - 1}{\alpha - 1} = \frac{p \alpha^n 1 - (\frac{1}{\alpha})^{n+1}}{1 - \frac{1}{\alpha}}$$

Note that: in here we used basic form:

$$\sum_{k=0}^{k=n} \alpha^{n-k} = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$
We can see clearly that when \( n \) go to infinite:

- if \( \alpha < 1 \), so the mean value stable
- if \( \alpha > 1 \), so the mean value unstable

### 2.2 Compute the \( \text{Var}[X[n]] \)

We have:

\[
\text{Var}[X[n]] = \text{Var} \left( \sum_{k=0}^{n} \alpha^{n-k} W[k] \right)
\]

But notice that this sequence independent, hence the var of a sum equal sum of var and if:

\[
Y = kX
\]

So:

\[
\text{Var}[Y] = k^2 \text{Var}[X]
\]

Then:

\[
\text{Var}[X[n]] = \sum_{k=0}^{n} \text{Var}(\alpha^{n-k} W[k]) = \sum_{k=0}^{n} \alpha^{2(n-k)} \text{Var}[W[k]]
\]

Note that: in here we used basic form:

\[
\sum_{k=0}^{n} \alpha^{2(n-k)} = \frac{\alpha^{2(n+1)} - 1}{\alpha^2 - 1}
\]

With \( \text{Bern}(p) \) distribution, we have:

\[
\text{Var} = p(1-p)
\]

Then:

\[
\text{Var}[X[n]] = p(1-p) \sum_{k=0}^{n} \alpha^{2(n-k)} = p(1-p) \alpha^{2n} \cdot \frac{1 - \frac{1}{\alpha^{2n+2}}}{1 - \frac{1}{\alpha}}
\]

### Example

We can compute:

\[
\]

But:

\[
E[\alpha W[0]^2] = \alpha E[W[0]^2] = \alpha(p(1-p) + p^2) = \alpha p
\]

\[
E[W[1]W[0]] = p^2
\]

Hence:

\[
E[X[1]X[0]] = \alpha p + p^2
\]
3 Poisson Distribution

Consider a random sequence consisting of iid RV: $\tau[n]$ (called tau in Greek language). For $n \geq 0$, each $\tau[n]$ is exponentially distributed. We can compute $\tau[n]$ as the form below:

$$f_{\tau[n]}(t) = \lambda e^{-\lambda t} u(t)$$

With:

$$u[t] = \begin{cases} 
0, & \text{with } t < 0 \\
1, & \text{with } t \geq 0 
\end{cases}$$

Note that $\tau[n]$ is meaning time between two consecutive event that follow Poisson distribution.

Let call:

$$T[n] = \sum_{k=0}^{k=n} \tau[k]$$

So, $T[n]$ here is a random sequence. For example, we can see a time waiting a bus follow Poisson Distribution. We can see the relationship between $\tau[k]$ and $T[n]$ in the below picture:

![Figure 4: Relate between $\tau[k]$ and $T[n]$](image)

And, using Erlang formulation (very common in communication when computing the time waiting in calling in PSTN network, you can see Erlang A,B,C in internet).
We have:

\[ f_{T[n]}(t) = \frac{(\lambda t)^{n-1}\lambda e^{-\lambda t}u(t)}{(n-1)!} \]  

(1)

See in picture we can compute:

\[ \tau[1] = T[1] \]

**Proof**

\[ f_{T[n]}(t) = f_{\tau[1]}(t) \ast f_{\tau[1]}(t) \ast \ldots f_{\tau[1]}(t) \]  

(2)

Since \( \tau[1], \tau[2], \ldots, \tau[n] \) are i.i.d. and \( T[n] = \sum_{k=1}^{n} \tau[k] \), we can write equation (2).

Now, we will use induction.

1) \( n=1 \), it’s true. From equation (1), we have

\[ f_{T[1]}(t) = \frac{(\lambda t)^{0}\lambda e^{-\lambda t}}{0!} = \lambda e^{-\lambda t} \]

where \( \lambda e^{-\lambda t} \) is exponential pdf of \( \tau[n] \).

2) Assume that

\[ f_{T[n-1]}(t) = \frac{(\lambda t)^{n-2}\lambda e^{-\lambda t}}{(n-2)!}u(t) \]

Then, we want to prove that

\[ f_{T[n]}(t) = \frac{(\lambda t)^{n-1}\lambda e^{-\lambda t}}{(n-1)!}u(t) \]

Now, from equation (2), we have
\( f_{\mathcal{W}[n]}(t) = f_{\mathcal{W}[n-1]}(t) \ast f_{\mathcal{W}[1]}(t) \)
\[
= \left[ \frac{(\lambda t)^{n-2} e^{-\lambda t}}{(n-2)!} \ast \lambda e^{-\lambda t} \right] u(t)
\]
\[
\text{(Since } f(t) \ast g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds \text{)}
\]
\[
= \int_{-\infty}^{\infty} \frac{\lambda s^{n-2} e^{-\lambda s}}{(n-2)!} \lambda e^{-\lambda(t-s)} u(s)u(t-s)ds
\]
\[
= \int_{0}^{t} \frac{\lambda s^{n-2} e^{-\lambda s}}{(n-2)!} \lambda e^{-\lambda(t-s)} ds
\]
\[
= \frac{\lambda^n e^{-\lambda t}}{(n-2)!} t^{n-1}
\]
\[
= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-2)! (n-1)}
\]

**Example** (pairwise average)

\( W[n] \) is an i.i.d. sequence with

\[
E[W[n]] = 0
\]
\[
R_W[k, l] = a^2 \delta[k - l] \quad (a \text{ is a constant})
\]

where

\[
\delta[n] \triangleq \begin{cases} 
1 & n=0 \\
0 & n\neq0
\end{cases}
\]


Compute \( R_{XX}[k, l] \)

\[
R_{XX}[k, l] = E[X[k]X[l]]
\]
\[
= E[(W[k] + W[k-1])(W[l] + W[l-1])]
\]
\[
\]
\[
= R_W(k, l) + R_W(k, l - 1) + R_W(k - 1, l) + R_W(k - 1, l - 1)
\]
\[
= a^2 \delta[k - l] + a^2 \delta[k - l + 1] + a^2 \delta[k - 1 - l] + a^2 \delta[k - l]
\]
\[
= 2a^2 \delta[k - l] + a^2 \delta[k - l + 1] + a^2 \delta[k - 1 - l]
\]

By \( R_{XX}[k, l] \), we have graph in figure 5.
Figure 5: Diagram of the tri-diagonal correlation function

What the graph say is that

\[ X[1] \text{ is uncorrelated with } X[3] \]
\[ X[2] \text{ is uncorrelated with } X[4] \]
\[ \vdots \]
\[ X[n] \text{ is uncorrelated with } X[n + 2] \]

We can actually see this intuitively

\[ \vdots \]

\( X[1] \) and \( X[3] \) are independent since they show no common \( W[\cdot] \).